Category Theory C2.7

Brian Chan

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0.1 Purpose

These notes are a summary of the material covered in the Category Theory course (C2.7) from the University of Oxford. They are primarily based on the lecture notes for the subject in [Saf23]. Additional useful references for the material in these notes and for basic category theory in general are [Bor94a], [Lei14] and [Rie17]. There are no strict prerequisites required to understand the notes. However, experience working in at least one category is helpful (for instance, the category of groups **Grp** via a first course in group theory).

Chapter 1

The basic language of category theory

1.1 Definition and basic examples

The concept of a category crops up in a multitude of different fields, ranging from group theory to algebraic topology (via 2-categories and higher order category theory). Certain constructions such as taking quotients or pullbacks in different contexts/categories have an elegant and unified description in category theory. Additionally, there are many instances of adjoint pairs of functors which appear in fields such as representation theory (induction and restriction) and multilinear algebra (Hom-tensor adjunction). To put it simply, category theory is **very powerful and pervasive**.

Definition 1.1.1. A **category** $\mathscr{C}$ is a triple consisting of:

1. A class of **objects** $\mathrm{ob}(\mathscr{C})$,
2. A class of **morphisms** (or arrows) between the objects $Hom(\mathscr{C})$. We say that the morphism $f : A \to B$ is an element of $Hom_{\mathscr{C}}(A, B)$, which denotes the class of all morphisms from A to B. In this case, A is deemed the **source object** and B is the **target object**.
3. A binary operation

$$\begin{array}{rccc} \circ: & Hom(B,C) \times Hom(A,B) & \to & Hom(A,C) \\ & (g,f) & \mapsto & g \circ f \end{array}$$

called **composition of morphisms**.

The binary operation of composition must satisfy the following two properties:

1. **Associativity**: $(f \circ g) \circ h = f \circ (g \circ h)$,
2. **Identity**: If A is an object in $\mathscr{C}$ then there exists a morphism $1_A : A \to A$ such that if $f \in Hom(A, B)$ and $g \in Hom(B, A)$ then $f \circ 1_A = f$ and $1_A \circ g = g$. The morphism 1_A is usually called the **identity morphism** on A.

Here are some basic examples of categories.

Example 1.1.1. The category **Set** is the category whose objects are sets and whose morphisms are functions between sets. In particular, the objects of **Set** raise the issue of Russell's paradox — there is no "set of all sets". Indeed, we were careful to say a "class of objects" and not a "set of objects" in the definition of a category. This observation segues into a philosophical discussion about the foundations of category theory which we will not pursue here. For further details, consult [Mur06] for a brief discussion. One way this observation is resolved is via *Grothendieck universes.*

Example 1.1.2. The category **Grp** is the category whose objects are groups and whose morphisms are group homomorphisms. The category **Ab** is the category whose objects are abelian groups and whose morphisms are group homomorphisms (between abelian groups). The category **Ab** is a subcategory of **Grp** — the objects/morphisms in **Ab** are also objects/morphisms in **Grp**.

Example 1.1.3. Here are examples of categories which are topological in nature. The category **Top** is the category whose objects are topological spaces and whose morphisms are continuous functions. The category **C*-Alg** is the category whose objects are C*-algebras and whose morphisms are *-homomorphisms.

Example 1.1.4. Let k be a field. The category k**-Vect** is the category whose objects are k-vector spaces and whose morphisms are linear maps. More generally, let R be a commutative ring. The category R**-Mod** is the category whose objects are R-modules and whose morphisms are R-module homomorphisms.

It is likely that the reader has already encountered at least one of the above examples of categories. As we progress, we will see many other examples of categories. Here is one easy way of constructing a category.

Definition 1.1.2. Let $\mathscr{C}$ be a category. The opposite category of $\mathscr{C}$, denoted by $\mathscr{C}^{op}$, is the category whose objects are the objects of $\mathscr{C}$ and whose morphisms are given by "reversing the morphisms". That is, if A, B are objects in $\mathscr{C}$ then

$$Hom_{\mathscr{C}^{op}}(A, B) = Hom_{\mathscr{C}}(B, A).$$

1.2 Types of morphisms in a category

Given a category, there are three basic types of morphisms.

Definition 1.2.1. Let $\mathscr{C}$ be a category. Suppose that we have the following diagram in $\mathscr{C}$:

$$U \xrightarrow{g} X \underset{h'}{\overset{h}{\rightrightarrows}} Y \xrightarrow{f} Z$$

We say that g **equalizes the pair** (h, h') if $h \circ g = h' \circ g$. Furthermore, f **coequalizes the pair** (h, h') if $f \circ h = f \circ h'$.

We say that f is a **monomorphism** when the only pairs (h, h') which are coequalized by f are pairs of the form (h, h). Dually, g is an **epimorphism** when the only pairs (h, h') which are equalized by g are pairs of the form (h, h).

To state it more explicitly, a morphism $f : X \to Y$ is a monomorphism if the following statement is satisfied: if $g, h \in Hom_{\mathscr{C}}(Z, X)$ and $f \circ g = f \circ h$ then $g = h$. Similarly, a morphism $f : X \to Y$ is an epimorphism if the following statement is satisfied: if $j, k \in Hom_{\mathscr{C}}(Y, U)$ and $j \circ f = k \circ f$ then $j = k$.

As a first example, we will characterise the epimorphisms and monomorphisms in **Set**.

Lemma 1.2.1. *In the category of sets **Set**, a function (a morphism of sets) $f : X \to Y$ is a monomorphism if and only if f is an injective function. Moreover, f is an epimorphism if and only if f is a surjective function.*

Proof. Assume that we have the following diagram in the category **Set**:

$$U \xrightarrow{g} X \underset{h'}{\overset{h}{\rightrightarrows}} Y \xrightarrow{f} Z$$

To show: (a) If g is surjective, then g is an epimorphism.

(b) If g is an epimorphism, then g is surjective.

(c) If f is injective, then f is a monomorphism.

(d) If f is a monomorphism, then f is injective.

(a) Assume that g is surjective and that $h \circ g = h' \circ g$. Since $g : U \to X$ is surjective then h and h' must agree on the image $g(U) = X$. So $h = h'$ and g is an epimorphism.

(b) We will prove this by contrapositive. Assume that g is not a surjective function. Then, there exists an element $x \in X$ such that $x \notin g(U)$. The key point here is that we can do anything with the element x. Define the functions $h, h' : X \to Y$ such that $h(x) \neq h'(x)$ and if $z \in X - \{x\}$ then $h(z) = h'(z)$. By construction, $h \neq h'$ but $h \circ g = h' \circ g$. Hence, the function g equalizes the pair (h, h') with $h \neq h'$, which shows that g is not an epimorphism as required.

(c) Assume that f is an injective function and that $f \circ h = f \circ h'$. If $x \in X$ then $f(h(x)) = f(h'(x))$ and since f is injective then $h(x) = h'(x)$. Thus, f is a monomorphism.

(d) We will again prove the contrapositive statement. Assume that f is not an injective function. Then, there exists $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $f(y_1) = f(y_2)$. Fix $x \in X$ and construct the functions $h, h' : X \to Y$ such that $h(x) = y_1$, $h'(x) = y_2$ and if $z \in X - \{x\}$ then $h(z) = h'(z)$. By construction, $h \neq h'$ but $f \circ h = f \circ h'$. Therefore, f coequalizes the pair of functions (h, h') with $h \neq h'$. So f is not a monomorphism. □

Example 1.2.1. In **Grp**, the monomorphisms are the injective group homomorphisms and the epimorphisms are the surjective group homomorphisms. The same statement holds for the subcategory **Ab**. See [Lei14, Example 5.1.31, Example 5.2.19].

The following properties of monomorphisms and epimorphisms under composition are proved by applying the definitions.

Theorem 1.2.2. *Let $\mathscr{C}$ be a category and f, g be morphisms in $\mathscr{C}$.*

1. *If f and g are monomorphisms in $\mathscr{C}$, then $g \circ f$ is also a monomorphism.*

2. *If $g \circ f$ is a monomorphism in $\mathscr{C}$, then f is also a monomorphism.*

3. *If f and g are epimorphisms in $\mathscr{C}$, then $g \circ f$ is also an epimorphism.*

4. *If $g \circ f$ is an epimorphism in $\mathscr{C}$, then g is also an epimorphism.*

The last type of morphism detects whether two objects in a category are the "same".

Definition 1.2.2. Let $\mathscr{C}$ be a category, A, B be objects in $\mathscr{C}$ and $f \in Hom_{\mathscr{C}}(A, B)$ be a morphism. We say that f is an **isomorphism** if there exists a morphism $f^{-1} \in Hom_{\mathscr{C}}(B, A)$ such that $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$.

Lemma 1.2.3. *Let $\mathscr{C}$ be a category, A, B be objects in $\mathscr{C}$ and $f : A \to B$ be an isomorphism. Then, f is a monomorphism and an epimorphism.*

Proof. Assume that $f : A \to B$ is a morphism in the category $\mathscr{C}$. Suppose that $h, h' : B \to C$ are morphisms such that $h \circ f = h' \circ f$. By precomposing with the inverse map f^{-1}, we find that $h \circ (f \circ f^{-1}) = h' \circ (f \circ f^{-1})$ and consequently, $h = h'$. So f is an epimorphism.

Now assume that $g, g' : Z \to A$ are morphisms such that $f \circ g = f \circ g'$. By composing with f^{-1}, we find that $g = g'$ and f is a monomorphism. □

Interestingly, the converse of Lemma 1.2.3 does not hold. The counterexample we will provide is in the category of monoids and can be found in [Awo10, Section 2].

Definition 1.2.3. The category **Mon** is the category whose objects are monoids (groups without inversion) and whose morphisms are monoid homomorphisms respecting the binary operation.

We will have to do some work in the category **Mon** in order to understand the counterexample. First, we will construct a particular monoid from an arbitrary set.

Definition 1.2.4. Let X be a set. The set of **words** on X, denoted by X^*, is the set whose elements are words (of finite length) whose letters are the elements of X. That is,

$$X^* = \{x_1 x_2 \dots x_n \mid n \in \mathbb{Z}_{\geq 0},\ x_i \in X\}.$$

If X is a set then X^* can be turned into a monoid by defining a binary operation which simply concatenates words.

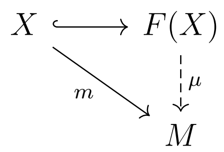
$$
\begin{array}{rcl} *: \quad X^* \times X^* & \to & X^* \\ (x, y) & \mapsto & xy. \end{array} \tag{1.1}
$$

One can show that the binary operation in equation (1.1) is associative and that the empty word (the word with no letters) is the unit associated to $*$. We denote the empty word by 1.

Definition 1.2.5. Let X be a set. The **free monoid** on X, denoted by $F(X)$, is the pair $(X^*, *)$ where $*$ is the binary operation in equation (1.1).

The point of constructing the free monoid is that it has the following universal property. We will introduce a piece of notation from [Awo10, Section 1.7] for what follows — if M is a monoid then we denote its underlying set by $|M|$.

Theorem 1.2.4. *Let X be a set. The free monoid $F(X)$ on X satisfies the following universal property: If M is a monoid and $m : X \to |M|$ is a function of sets then there exists a unique monoid homomorphism $\mu : F(X) \to M$ such that the following diagram in **Mon** commutes:*



Proof. Assume that X is a set and $F(X)$ is the free monoid on X. Then, we have the inclusion map of sets $\iota : X \hookrightarrow |F(X)|$. Assume that $m : X \to |M|$ is a function of sets. Define the map μ by

$$
\begin{array}{rccc} \mu : & F(X) & \to & M \\ & x_1 \dots x_n & \mapsto & m(x_1)m(x_2)\dots m(x_n) \\ & 1 & \mapsto & 1_M \end{array}
$$

Here 1_M is the unit of M and we recall that 1 is the empty word in $F(X)$. By construction, μ is a monoid homomorphism and it is straightforward to check that $m = \mu \circ \iota$. To see that μ is unique, assume that $\phi : F(X) \to M$ is another monoid homomorphisms satisfying $m = \phi \circ \iota$. If $x \in X$ then

$$
(\phi \circ \iota)(x) = \phi(x) = m(x) = (\mu \circ \iota)(x) = \mu(x).
$$

Therefore, $\phi = \mu$ and μ is unique. □

Now we will use the universal property of the free monoid to give a characterisation of the monomorphisms in **Mon**.

Theorem 1.2.5. *Let M, N be monoids and $f : M \to N$ be a monoid homomorphism. Then f is a monomorphism if and only if the underlying function of sets $|f| : |M| \to |N|$ is a monomorphism.*

Proof. Assume that M and N are monoids and $f : M \to N$ is a monoid homomorphism. Let $|f| : |M| \to |N|$ be the underlying function of sets. To be clear, if $m \in M$ then $|f|(m) = f(m)$. The map $|f|$ behaves exactly the same as f, but without the monoid structure on both sides.

To show: (a) If $|f|$ is a monomorphism in **Set** then f is a monomorphism in **Mon**.

(b) If f is a monomorphism in **Mon** then $|f|$ is a monomorphism in **Set**.

(a) Assume that $|f|$ is a monomorphism in **Set**. Assume that $g, h : P \to M$ are distinct monoid homomorphisms. Since $|f|$ is a monomorphism then $|f| \circ |g|$ and $|f| \circ |h|$ are distinct morphisms in **Set**. So

$$|f \circ g| = |f| \circ |g| \neq |f| \circ |h| = |f \circ h|$$

and consequently, $f \circ g \neq f \circ h$. Therefore f is a monomorphism in **Mon**.

(b) Assume that f is a monomorphism in **Mon**. By Lemma 1.2.1, it suffices to show that $|f|$ is an injective function. To this end, assume that x, y are distinct elements in $|M|$. We have two functions of sets given by

$$\begin{array}{rccc} \iota_x : & \{*\} & \to & M \\ & * & \mapsto & x \end{array}$$

and ι_y which is defined similarly. By the universal property of the free monoid in Theorem 1.2.4, there exist unique monoid homomorphisms $\phi_x : F(\{*\}) \to M$ and $\phi_y : F(\{*\}) \to M$ such that $\phi_x(*) = x$ and $\phi_y(*) = y$. Now $\phi_x \neq \phi_y$ by construction and since f is a monomorphism in **Mon** then $f \circ \phi_x \neq f \circ \phi_y$. So

$$f(x) = (f \circ \phi_x)(*) \neq (f \circ \phi_y)(*) = f(y)$$

and subsequently, $|f|$ is an injective function. This completes the proof. □

We are now able to give a counterexample to the converse of Lemma 1.2.3.

Example 1.2.2. Let ι denote the monoid homomorphism

$$\begin{array}{rccc} \iota : & \mathbb{Z}_{\geq 0} & \to & \mathbb{Z} \\ & x & \mapsto & x. \end{array}$$

To be clear, the binary operations on both sides are addition. By Theorem 1.2.5, ι is a monomorphism in **Mon**. To see that ι is an epimorphism in **Mon**, let $g, h : \mathbb{Z} \to M$ be monoid homomorphisms such that $g \circ \iota = h \circ \iota$. If $m \in \mathbb{Z}_{\geq 0}$ then $g(m) = h(m)$. Observe that

$$g(-m) = \sum_{i=1}^{m} g(-1).$$

Hence, it suffices to show that $g(-1) = h(-1)$. Let 1_M denote the identity element of M. We compute directly that

$$\begin{aligned} g(-1) &= g(-1)1_M \\ &= g(-1)h(0) \\ &= g(-1)h(1)h(-1) \\ &= g(-1)g(1)h(-1) \\ &= g(0)h(-1) \\ &= 1_M h(-1) = h(-1). \end{aligned}$$

We conclude that $g = h$ on $\mathbb{Z}$. Therefore ι is an epimorphism in **Mon**. Finally, suppose for the sake of contradiction that ι is an isomorphism in **Mon**. Then there exists a monoid homomorphism $\mu : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ such that $\iota \circ \mu = id_{\mathbb{Z}}$ and $\mu \circ \iota = id_{\mathbb{Z}_{\geq 0}}$. By definition of ι,

$$-1 = (\iota \circ \mu)(-1) = \mu(-1) \in \mathbb{Z}_{\geq 0}$$

which is a blatant contradiction. Therefore, ι is not an isomorphism in **Mon** and supplies the desired counterexample to the converse of Lemma 1.2.3.

We briefly remark that the free monoid on a set we discussed in this section is a special case of a more general phenomenon in category theory — pairs of adjoint functors. We will study adjoint functors later on in these notes.

1.3 Functors

Analogously to morphisms in a category, we can define the notion of a "morphism between categories".

Definition 1.3.1. Let $\mathscr{C}, \mathscr{D}$ be categories. A **(covariant) functor** $F : \mathscr{C} \to \mathscr{D}$ is a map which satisfies the following properties:

1. If $C \in ob(\mathscr{C})$, then $F(C) \in ob(\mathscr{D})$,
2. If $C \in ob(\mathscr{C})$, then $F(id_C) = id_{F(C)}$, where id_C and $id_{F(C)}$ are the identity morphisms on C and $F(C)$ respectively,
3. If $X, Y, Z \in ob(\mathscr{C})$, $f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$, then $F(g \circ f) = F(g) \circ F(f)$.

Like the morphisms in well-known categories, a functor preserves the essential structures of a category — the identity morphism on every object and the composition operation. Usually, we simply refer to a covariant functor as a functor.

Example 1.3.1. Let G be a group and

$$[G, G] = \{[g, h] = ghg^{-1}h^{-1} \mid g, h \in G\}$$

be the commutator subgroup of G. The quotient $G^{ab} = G/[G, G]$ is the abelianisation of G. We also have the projection map $\pi_G : G \to G^{ab}$, which is a group morphism.

The functor $ab : \mathbf{Grp} \to \mathbf{Ab}$ sends a group G to its abelianisation G^{ab} and a group morphism $f : G \to H$ to the group morphism $f^{ab} : G^{ab} \to H^{ab}$. The morphism f^{ab} is the unique group morphism which makes the below diagram commute by the universal property of the quotient:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \pi_G}\downarrow & & \downarrow{\scriptstyle \pi_H} \\ G^{ab} & \underset{f^{ab}}{\dashrightarrow} & H^{ab} \end{array}$$

Definition 1.3.2. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **small** if the classes $ob(\mathscr{C})$ and $Hom(\mathscr{C})$ are actually sets.

Example 1.3.2. Here is an example of a small category. Let G be a group. If $g \in G$ then g induces a group homomorphism

$$\begin{array}{rccc} C_g : & G & \to & G \\ & h & \mapsto & ghg^{-1}. \end{array}$$

So G can be regarded as a small category with

$$ob(G) = \{G\} \qquad \text{and} \qquad Hom(G) = \{C_g : G \to G \mid g \in G\}.$$

Using the notion of a small category, we can really make the notion of a functor being a morphism between categories a reality with our next example of a category.

Example 1.3.3. The category **Cat** is the category whose objects are small categories and whose morphisms are functors between small categories.

Let us provide some more examples of functors.

Example 1.3.4. Let $U : \mathbf{Mon} \to \mathbf{Set}$ be defined in the following manner: if M is a monoid then $U(M) = |M|$ is the underlying set (just the monoid M but regarded as a set). If $f : M \to N$ is a monoid morphism then $U(f)$ is the underlying function of sets. The functor U forgets the monoid structure on the objects and morphisms in **Mon**. Such functors are called **forgetful functors**.

Example 1.3.5. Here is a well-known example of a functor from algebraic topology. The category $\mathbf{Top}_*$ is the category whose objects are pointed topological spaces (X, x_0) where $x_0 \in X$ is a chosen point (called the basepoint) and whose morphisms are basepoint-preserving continuous functions. Define a map π_1 from $\mathbf{Top}_*$ to **Grp** by

$$\begin{array}{rccc} \pi_1 : & \mathbf{Top}_* & \to & \mathbf{Grp} \\ & (X, x_0) & \mapsto & \pi_1(X, x_0) \\ & f : (X, x_0) \to (Y, y_0) & \mapsto & \pi_1(f) \end{array}$$

where if $f : (X, x_0) \to (Y, y_0)$ is a morphism in $\mathbf{Top}_*$ then $\pi_1(f)$ is the group morphism

$$\begin{array}{rccc} \pi_1(f) : & \pi_1(X, x_0) & \to & \pi_1(Y, y_0) \\ & [\gamma] & \mapsto & [f \circ \gamma] \end{array}$$

where $\gamma : [0, 1] \to X$ is a loop at the basepoint x_0 — that is, γ is a continuous function satisfying $\gamma(0) = \gamma(1) = x_0$. See [Bre93, Section 3.2] for an introduction to the fundamental group of a pointed topological space.

Example 1.3.6. Let $k \in \mathbb{Z}_{>0}$ and R be a commutative ring. Define

$$\begin{array}{rccc} \bigwedge^k : & R\text{-}\mathbf{Mod} & \to & R\text{-}\mathbf{Mod} \\ & M & \mapsto & \bigwedge^k(M) \\ & \phi : M \to N & \mapsto & \big(m_1 \wedge \cdots \wedge m_k \mapsto \phi(m_1) \wedge \cdots \wedge \phi(m_k)\big). \end{array}$$

Here $\bigwedge^k(M)$ is the k^{th} exterior power of M and is defined by

$$\bigwedge^k(M) = \{m_1 \wedge \cdots \wedge m_k \mid m_1, \ldots, m_k \in M\}.$$

Here, $\wedge$ is the wedge product. If M is a R-module and $m, n \in M$ then the wedge product satisfies the well-known anticommutative property $m \wedge n = -(n \wedge m)$. There is some work required to show that $\bigwedge^k$ is a well-defined functor. See [Rot03, Section 9.8] and [Rot03, Proposition 9.135] in particular.

We have a notion of functors which reverse the order of composition.

Definition 1.3.3. Let $\mathscr{C}, \mathscr{D}$ be categories. A **contravariant functor** $F : \mathscr{C} \to \mathscr{D}$ is a map which satisfies the following properties:

1. If $C \in ob(\mathscr{C})$, then $F(C) \in ob(\mathscr{D})$,
2. If $C \in ob(\mathscr{C})$, then $F(id_C) = id_{F(C)}$, where id_C and $id_{F(C)}$ are the identity morphisms on C and $F(C)$ respectively,
3. If $X, Y, Z \in ob(\mathscr{C})$, $f \in Hom(X, Y)$ and $g \in Hom(Y, Z)$, then $F(g \circ f) = F(f) \circ F(g)$.

A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ can be simply thought of as a covariant functor $F : \mathscr{C}^{op} \to \mathscr{D}$. Here is a specific case of an important class of contravariant functors called *presheaves*.

Example 1.3.7. Let (X, τ) be a topological space where X is a Riemann surface and τ is the manifold topology on X. Let $\mathscr{C}$ be the category whose objects are open subsets of X and whose morphisms are given explicitly by

$$Hom_{\mathscr{C}}(U, V) = \begin{cases} \{\iota_U^V : U \hookrightarrow V\}, \text{ if } U \subseteq V, \\ \emptyset, \text{ otherwise.} \end{cases}$$

Here, if $U \subseteq V$ then ι_U^V is the inclusion of U into V. Define

$$\begin{array}{rccc} \mathcal{O}_X : & \mathscr{C} & \to & \mathbf{Ab} \\ & U & \mapsto & \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\} \\ & \iota_U^V & \mapsto & res_U^V : \mathcal{O}_X(V) \to \mathcal{O}_X(U). \end{array}$$

where if U and V are open subsets of X satisfying $U \subseteq V$ then res_U^V is the restriction map. That is, if $f \in \mathcal{O}_X(V)$ then $res_U^V(f) = f|_U$.

Note that $\mathcal{O}_X$ is well-defined on objects of $\mathscr{C}$ because the set of holomorphic functions on an open subset U of X forms a ring with addition and multiplication defined pointwise. It is not too difficult to check that $\mathcal{O}_X$ is a contravariant functor, referred to as the **presheaf of holomorphic functions on** X. See [For81, §6] for an introduction to presheaves and sheaves in the context of Riemann surfaces.

Here are types of functors we will need later on.

Definition 1.3.4. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $H : \mathscr{C} \to \mathscr{D}$ be a functor. If X and Y are objects in $\mathscr{C}$ then H induces the map

$$H_{X,Y} : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(H(X), H(Y))$$

We say that H is a **faithful functor** if the following statement is satisfied: if X and Y are object in $\mathscr{C}$ then the map $H_{X,Y}$ is injective.

We say that H is a **full functor** if the following statement is satisfied: if X and Y are object in $\mathscr{C}$ then the map $H_{X,Y}$ is surjective.

We say that H is a **fully faithful functor** if the following statement is satisfied: if X and Y are object in $\mathscr{C}$ then the map $H_{X,Y}$ is bijective.

Example 1.3.8. Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor, which maps a group to its underlying set and a group morphism to the underlying function between the sets. We claim that U is a faithful functor. Let G, H be groups. To see why the mapping

$$U_{G,H} : Hom_{\mathbf{Grp}}(G, H) \to Hom_{\mathbf{Set}}(U(G), U(H))$$

is injective, suppose that $\phi_1, \phi_2 \in Hom_{\mathbf{Grp}}(G, H)$ and $U_{G,H}(\phi_1) = U_{G,H}(\phi_2)$. By definition of the forgetful functor U, ϕ_1 and ϕ_2 must agree on the underlying set $U(G)$ and hence, on G itself. So, $\phi_1 = \phi_2$ and $U_{G,H}$ must be injective. This demonstrates that U is a faithful functor.

On the other hand, to see that U is not a full functor, let e_H denote the identity element of the group H. Fix $h \in H - \{e_H\}$ and define the morphisms of sets

$$\begin{array}{rccc} \alpha : & U(G) & \to & U(H) \\ & g & \mapsto & h. \end{array}$$

Since $\alpha(e_G) \neq e_H$ then α is not a group homomorphism from G to H. So, $U_{G,H}$ is not surjective and consequently U is not full.

By using Lemma 1.2.1, we can identify a particular class of monomorphisms and epimorphisms in a category $\mathscr{C}$, provided that we have a faithful functor $F : \mathscr{C} \to \mathbf{Set}$.

Theorem 1.3.1. *Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to$ **Set** be a faithful functor. Let $f : A \to B$ be a morphism in $\mathscr{C}$.*

1. *If $F(f)$ is a monomorphism in **Set** (an injective map) then f is a monomorphism.*

2. *If $F(f)$ is an epimorphism in **Set** (a surjective map) then f is an epimorphism.*

Proof. Assume that $\mathscr{C}$ is a category and $F : \mathscr{C} \to \mathbf{Set}$ is a faithful functor. Assume that $f : A \to B$ is a morphism in $\mathscr{C}$ such that $F(f)$ is a monomorphism in **Set**. If C, D are objects in $\mathscr{C}$ then

$$F_{C,D} : Hom_{\mathscr{C}}(C, D) \to Hom_{\mathbf{Set}}(F(C), F(D))$$

is injective. To see that f is a monomorphism, assume that $g, h : C \to A$ are morphisms in $\mathscr{C}$ satisfying $f \circ g = f \circ h$. Then $F(f) \circ F(g) = F(f) \circ F(h)$ and since $F(f)$ is a monomorphism then $F(g) = F(h)$ in $Hom_{\mathbf{Set}}(F(C), F(D))$. Since $F_{C,D}$ is injective then $g = h$. Hence f is a monomorphism in $\mathscr{C}$.

Now assume that $F(f)$ is an epimorphism in **Set**. To see that f is an epimorphism, assume that $p, q : B \to D$ are morphisms in $\mathscr{C}$ satisfying $p \circ f = q \circ f$. Then $F(p) \circ F(f) = F(q) \circ F(f)$ and since $F(f)$ is an epimorphism then $F(p) = F(q)$ in $Hom_{\mathbf{Set}}(F(C), F(D))$. As before, $g = h$ and f is an epimorphism in $\mathscr{C}$. □

Example 1.3.9. As the previous example shows, the forgetful functor **Grp** $\to$ **Set** is faithful. Theorem 1.3.1 tells us that injective group morphisms are monomorphisms in **Grp** and that surjective group morphisms are epimorphisms in **Grp**. Recall from Example 1.2.1 that in fact, the monomorphisms in **Grp** are exactly the injective group morphisms and the epimorphisms in **Grp** are exactly the surjective group morphisms.

1.4 Natural transformations

Natural transformations can be thought of as maps between functors. The adjective “natural” refers to the fact that natural transformations between functors behave nicely when both functors are applied to a morphism.

Definition 1.4.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** $\alpha : F \Rightarrow G$ is a family of morphisms

$$\{\alpha_A : F(A) \to G(A) \mid A \in \mathscr{C}\}$$

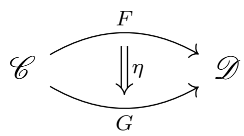
such that if $f : A \to A'$ is a morphism in $\mathscr{C}$ then the following diagram in $\mathscr{D}$ commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ G(A) & \xrightarrow[G(f)]{} & G(A') \end{array}$$

If the α_A are all isomorphisms in $\mathscr{D}$, then α is said to be a **natural isomorphism**.

We say that the functors F and G are **naturally isomorphic** if there exists a natural isomorphism $\alpha : F \Rightarrow G$. We will use the notation $F \simeq G$ to denote that F is naturally isomorphic to G.

Diagrammatically, natural transformations $\eta : F \Rightarrow G$ are represented by



Example 1.4.1. Recall the abelianisation functor ab : **Grp** $\to$ **Ab** from Example 1.3.1. If G is a group then let $\pi_G : G \to G^{ab}$ denote the quotient group morphism. Then the collection

$$\{\pi_G : G \to G^{ab} \mid G \text{ is a group}\}$$

defines a natural transformation from the identity functor id : **Grp** $\to$ **Grp** to ab. This is due to the commutative square in Example 1.3.1.

The following example of a category highlights the notion that natural transformations are morphisms of functors.

Example 1.4.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is the category whose objects are functors $F : \mathscr{C} \to \mathscr{D}$ and whose morphisms are natural transformations $\eta : F \Rightarrow G$, where F and G are both functors from $\mathscr{C}$ to $\mathscr{D}$. The isomorphisms in $\mathcal{F}(\mathscr{C}, \mathscr{D})$ are the natural isomorphisms. A category of the form $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is evidently called a **functor category**.

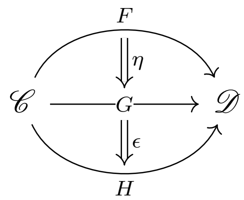
Just like functors, natural transformations can be composed. However, there are two different types of compositions.

Vertical composition:

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let F, G, H be functors from $\mathscr{C}$ to $\mathscr{D}$ and $\eta : F \Rightarrow G$ and $\epsilon : G \Rightarrow H$ be natural transformations. One can check that the collection of morphisms

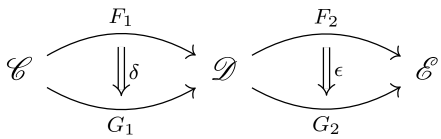
$$\{\epsilon_A \circ \eta_A : F(A) \to H(A) \mid A \in \mathbf{ob}(\mathscr{C})\}$$

defines a natural transformation $\epsilon \circ \eta : F \Rightarrow H$. This composition is called vertical due to the following diagram:



Horizontal composition:

Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories. Let $F_1, G_1 : \mathscr{C} \to \mathscr{D}$ and $F_2, G_2 : \mathscr{D} \to \mathscr{E}$ be functors. Let $\delta : F_1 \Rightarrow G_1$ and $\epsilon : F_2 \Rightarrow G_2$ be natural transformations.



(1.2)

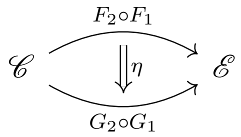
Then the collection of morphisms in $\mathscr{E}$

$$\{\epsilon_{G_1(A)} \circ F_2(\delta_A) : (F_2 \circ F_1)(A) \to (G_2 \circ G_1)(A) \mid A \in \mathbf{ob}(\mathscr{C})\}$$

defines a natural transformation $\eta : F_2 \circ F_1 \Rightarrow G_2 \circ G_1$. To be clear, $F_2(\delta_A)$ is a morphism from $(F_2 \circ F_1)(A)$ to $(F_2 \circ G_1)(A)$ and $\epsilon_{G_1(A)}$ is a morphism from $(F_2 \circ G_1)(A)$ to $(G_2 \circ G_1)(A)$. One can also do things in the opposite order and take the collection of morphisms

$$\{G_2(\delta_A) \circ \epsilon_{F_1(A)} : (F_2 \circ F_1)(A) \to (G_2 \circ G_1)(A) \mid A \in \mathbf{ob}(\mathscr{C})\}.$$

It turns out that by the commutative diagram in Definition 1.4.1, this still yields the same natural transformation $\eta : F_2 \circ F_1 \Rightarrow G_2 \circ G_1$ as before. This type of composition is called horizontal because the natural transformations in Diagram (1.2) have been composed and condensed into the single horizontal diagram



1.5 Equivalence of categories

How do we say that two categories are isomorphic or equivalent? Thinking of morphisms as categories, one might define an equivalence of categories in a similar vein to an isomorphism. However in category theory, we do not care about the individual identities of our objects. We only care if they are isomorphic to each other or not. Extending this line of thinking to categories, we make the following definition

Definition 1.5.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. An **equivalence of categories** between $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ together with a pair of natural isomorphisms $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$. Here, $id_{\mathscr{C}}$ and $id_{\mathscr{D}}$ are identity functors on $\mathscr{C}$ and $\mathscr{D}$ respectively.

Example 1.5.1. As an example of an equivalence of categories, let k be a field and $\textbf{Fin-Vect}_k$ be the category of finite dimensional k-vector spaces. Define the map

$$\begin{array}{rccc} D: & \textbf{Fin-Vect}_k^{op} & \to & \textbf{Fin-Vect}_k \\ & V & \mapsto & V^* = Hom_k(V,k) \\ & f: V \to W & \mapsto & f^*: W^* \to V^*. \end{array} \tag{1.3}$$

For clarity, if $\alpha : W \to k$ is a k-linear functional then $f^*(\alpha) = \alpha \circ f$. It is straightforward to check that D is a (contravariant) functor. We claim that D is an equivalence of categories. The composite $D \circ D$ is explicitly

$$\begin{array}{rccc} D \circ D: & \textbf{Fin-Vect}_k^{op} & \to & \textbf{Fin-Vect}_k^{op} \\ & V & \mapsto & (V^*)^* = Hom_k(V^*,k) \\ & f: V \to W & \mapsto & (f^*)^*: (V^*)^* \to (W^*)^*. \end{array}$$

In equation (1.3), we can also consider D as a functor from $\textbf{Fin-Vect}_k$ to $\textbf{Fin-Vect}_k^{op}$. If V is a finite dimensional k-vector space then define $\eta_V : V \to (D \circ D)(V)$ by

$$\begin{array}{rccc} \eta_V: & V & \to & (D \circ D)(V) = (V^*)^* \\ & v & \mapsto & ev_v \end{array}$$

where $ev_v : V^* \to k$ maps $f \in V^*$ to $f(v) \in k$. It is again easy to check that η_V is a linear transformation from V to $(V^*)^*$.

To show: (a) η is a natural isomorphism from $id_{\textbf{Fin-Vect}_k^{op}}$ to $D \circ D$.

(a) Assume that $f : V \to W$ is a morphism in $\textbf{Fin-Vect}_k^{op}$. We compute directly that if $v \in V$ and $\beta \in W^*$ then

$$ev_{f(v)}(\beta) = \beta(f(v)) = (\beta \circ f)(v) = f^*(\beta)(v) = (ev_v \circ f^*)(\beta)$$

and

$$\begin{aligned}
(\eta_W \circ id_{\textbf{Fin-Vect}_k^{op}}(f))(v) &= (\eta_W \circ f)(v) \\
&= \eta_W(f(v)) = ev_{f(v)} \\
&= ev_v \circ f^* \\
&= (f^*)^*(ev_v) \\
&= (D \circ D)(f)(ev_v) \\
&= ((D \circ D)(f) \circ \eta_V)(v).
\end{aligned}$$

Hence, η is a natural transformation from $id_{\textbf{Fin-Vect}_k^{op}}$ to $D \circ D$. To see that η is a natural isomorphism, it suffices to show that $\eta_V : V \to (V^*)^*$ is injective.

Assume that $v_1, v_2 \in V$ satisfy $\eta_V(v_1) = \eta_V(v_2)$. Then, $ev_{v_1} = ev_{v_2}$, which means that if $f \in V^*$ then $f(v_1) = f(v_2)$. By linearity of f, we find that

$$v_1 - v_2 \in \bigcap_{f \in V^*} \ker f.$$

So, $v_1 - v_2 = 0$ and $v_1 = v_2$. We conclude that if V is a finite dimensional k-vector space then η_V is an injective linear transformation between two finite dimensional k-vector spaces. Hence, it is a vector space isomorphism and η is a natural isomorphism.

By a similar argument where we interchange the roles of $\textbf{Fin-Vect}_k$ and $\textbf{Fin-Vect}_k^{op}$, we also find that η_V defines a natural isomorphism from $id_{\textbf{Fin-Vect}_k}$ to $D \circ D$. Consequently, D, D, η_V and η_V define an equivalence of categories between $\textbf{Fin-Vect}_k$ and $\textbf{Fin-Vect}_k^{op}$.

Definition 1.5.1 is often not useful in practice because constructing the required functor G is usually difficult. The main result of this section is to prove a useful characterisation of equivalences of categories. First, we need the following definition.

Definition 1.5.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F is **essentially surjective** if the following statement is satisfied: If Y is an object in $\mathscr{D}$ then there exists an object X in $\mathscr{C}$ such that $F(X) \cong Y$.

Here is the main result on equivalence of categories.

Theorem 1.5.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The following are equivalent:*

1. *There exists an equivalence of categories between $\mathscr{C}$ and $\mathscr{D}$ consisting of functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$, $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$.*

2. *There exists a fully faithful essentially surjective functor $H : \mathscr{C} \to \mathscr{D}$.*

3. *There exists a fully faithful functor $K : \mathscr{C} \to \mathscr{D}$, a functor $L : \mathscr{D} \to \mathscr{C}$ and a natural isomorphism $\delta : id_{\mathscr{D}} \Rightarrow K \circ L$.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories.

The third statement implies the second:

Assume that $K : \mathscr{C} \to \mathscr{D}$ is a fully faithful functor, $L : \mathscr{D} \to \mathscr{C}$ is a functor and $\delta : id_{\mathscr{D}} \Rightarrow K \circ L$ is a natural isomorphism. We claim that K is essentially surjective. Assume that D is an object in $\mathscr{D}$. Since η is a natural isomorphism then η_D is an isomorphism from D to $(K \circ L)(D)$. So $D \cong (K \circ L)(D) = K(L(D))$. Therefore K is fully faithful and essentially surjective.

The second statement implies the first:

Assume that $H : \mathscr{C} \to \mathscr{D}$ is a fully faithful essentially surjective functor. We will first build a functor $F : \mathscr{D} \to \mathscr{C}$. If D is an object in $\mathscr{D}$ then there exists an object $C_D \in \mathscr{C}$ such that $H(C_D) \cong D$ because H is essentially surjective. Let $\varphi_D : D \to H(C_D)$ denote the isomorphism between D and $H(C_D)$. On the objects D of $\mathscr{D}$, we define $F(D) = C_D$.

Next, we have to define F on morphisms in $\mathscr{D}$. Let $g : D \to D'$ be a morphism in $\mathscr{D}$. Since H is fully faithful then there exists a *unique* morphism $f : F(D) \to F(D')$ such that the following diagram in $\mathscr{D}$ commutes:

$$
\begin{array}{ccc}
D & \xrightarrow{\quad g \quad} & D' \\
{\scriptstyle \varphi_D}\downarrow & & \downarrow{\scriptstyle \varphi_{D'}} \\
(H \circ F)(D) & \xrightarrow[H(f)]{} & (H \circ F)(D')
\end{array}
$$

Subsequently, we define $F(g) = f$. To see that F defines a functor from $\mathscr{D}$ to $\mathscr{C}$, we first note that the following square commutes in $\mathscr{D}$:

$$
\begin{array}{ccc}
D & \xrightarrow{\quad id_D \quad} & D \\
{\scriptstyle \varphi_D}\downarrow & & \downarrow{\scriptstyle \varphi_D} \\
(H \circ F)(D) & \xrightarrow[id_{(H \circ F)(D)}]{} & (H \circ F)(D)
\end{array}
$$

Note that $id_{(H \circ F)(D)} = H(id_{F(D)})$. By construction of F, $H(F(id_D))$ also makes the above diagram commute. By uniqueness, we must have $H(F(id_D)) = H(id_{F(D)})$. Since H is a faithful functor, we deduce that $F(id_D) = id_{F(D)}$.

For composition of morphisms, suppose that $\alpha \in Hom_{\mathscr{D}}(D, D')$ and $\beta \in Hom_{\mathscr{D}}(D', D'')$. Then the following diagram in $\mathscr{D}$ commutes:

$$
\begin{array}{ccc}
D & \xrightarrow{\quad \beta \circ \alpha \quad} & D'' \\
{\scriptstyle \varphi_D}\downarrow & & \downarrow{\scriptstyle \varphi_{D''}} \\
(H \circ F)(D) & \xrightarrow[H(F(\beta \circ \alpha))]{} & (H \circ F)(D'')
\end{array}
$$

But, we also have the following commutative diagram in $\mathscr{D}$:

$$
\begin{array}{ccccc}
D & \xrightarrow{\quad \alpha \quad} & D' & \xrightarrow{\quad \beta \quad} & D'' \\
\downarrow{\scriptstyle \varphi_D} & & \downarrow{\scriptstyle \varphi_{D'}} & & \downarrow{\scriptstyle \varphi_{D''}} \\
(H \circ F)(D) & \xrightarrow[H(F(\alpha))]{} & (H \circ F)(D') & \xrightarrow[H(F(\beta))]{} & (H \circ F)(D'')
\end{array}
$$

Therefore $H(F(\beta) \circ F(\alpha))$ and $H(F(\beta \circ \alpha))$ both make the same diagram commute. By uniqueness, $H(F(\beta \circ \alpha)) = H(F(\beta) \circ F(\alpha))$ and since H is faithful, we thus have $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. Hence $F : \mathscr{D} \to \mathscr{C}$ is a functor and $\varphi : id_{\mathscr{D}} \Rightarrow H \circ F$ is a natural isomorphism.

It remains to construct a natural isomorphism from $id_{\mathscr{C}}$ to $F \circ H$. Let C be an object in $\mathscr{C}$. Then, $\varphi_{H(C)}$ is an isomorphism between $H(C)$ and $(H \circ F \circ H)(C)$. Now H is fully faithful. So there exists a unique morphism $\psi_C : C \to (F \circ H)(C)$ such that $H(\psi_C) = \varphi_{H(C)}$. However, $\varphi_{H(C)}$ is an isomorphism. So, there exists a morphism $\gamma : (H \circ F \circ H)(C) \to H(C)$ such that

$$\gamma \circ \varphi_{H(C)} = id_{H(C)} \qquad \text{and} \qquad \varphi_{H(C)} \circ \gamma = id_{(H\circ F\circ H)(C)}.$$

Using the fact that H is fully faithful again, there exists a unique morphism $\delta : (F \circ H)(C) \to C$ such that $H(\delta) = \gamma$. Now observe that

$$\begin{aligned} H(\delta \circ \psi_C) &= H(\delta) \circ H(\psi_C) \\ &= \gamma \circ \varphi_{H(C)} \\ &= id_{H(C)} = H(id_C) \end{aligned}$$

and similarly, $H(\psi_C \circ \delta) = H(id_{(F\circ H)(C)})$. Since H is faithful then $\delta \circ \psi_C = id_C$ and $\psi_C \circ \delta = id_{(F\circ H)(C)}$. We conclude that if C is an object in $\mathscr{C}$ then ψ_C is an isomorphism.

To see that ψ is a natural isomorphism between $id_{\mathscr{C}}$ and $F \circ H$, consider the following diagram in $\mathscr{C}$:

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C' \\ {\scriptstyle \psi_C}\downarrow & & \downarrow{\scriptstyle \psi_{C'}} \\ (F\circ H)(C) & \xrightarrow[(F\circ H)(f)]{} & (F\circ H)(C') \end{array}$$

By applying H to this diagram, we obtain a commutative diagram in $\mathscr{D}$. Since H is faithful, we deduce that the above diagram in $\mathscr{C}$ commutes. Hence, $\psi : id_{\mathscr{C}} \Rightarrow F \circ H$ is a natural isomorphism and (H, F, φ, ψ) defines an equivalence of categories between $\mathscr{C}$ and $\mathscr{D}$.

The first statement implies the third:

Assume that there exists a pair of functors $H : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{D} \to \mathscr{C}$ and a pair of natural isomorphisms $\eta : id_{\mathscr{C}} \Rightarrow F \circ H$ and $\epsilon : H \circ F \Rightarrow id_{\mathscr{D}}$. It suffices to show that H is fully faithful. Assume that X, Y are objects in $\mathscr{C}$. Then, the functor H induces the map

$$H_{X,Y} : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(H(X), H(Y)).$$

To see that H is faithful, we must show that $H_{X,Y}$ is an injective map. Since η is a natural isomorphism then the function

$$(F \circ H)_{X,Y} : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{C}}((F \circ H)(X), (F \circ H)(Y))$$

is bijective. Observe that $(F \circ H)_{X,Y}$ is the composite $F_{H(X),H(Y)} \circ H_{X,Y}$. Since $(F \circ H)_{X,Y}$ is injective then $H_{X,Y}$ must also be injective.

To see that H is fully faithful, the natural isomorphism $\epsilon : H \circ F \Rightarrow id_{\mathscr{D}}$ tells us that the following induced map is bijective:

$$Hom_{\mathscr{D}}(H(X), H(Y)) \to Hom_{\mathscr{D}}((H \circ F \circ H)(X), (H \circ F \circ H)(Y))$$

But this map is the composite $H_{(F \circ H)(X),(F \circ H)(Y)} \circ F_{H(X),H(Y)}$. Arguing in a similar manner to before, we deduce that the map $H_{(F \circ H)(X),(F \circ H)(Y)}$ is surjective. Since H is also faithful then $H_{(F \circ H)(X),(F \circ H)(Y)}$ defines the bijection

$$Hom_{\mathscr{C}}((F \circ H)(X), (F \circ H)(Y)) \cong Hom_{\mathscr{D}}((H \circ F \circ H)(X), (H \circ F \circ H)(Y)).$$

But the LHS is isomorphic to $Hom_{\mathscr{C}}(X, Y)$, whereas the RHS is isomorphic to $Hom_{\mathscr{D}}(H(X), H(Y))$. Thus, $Hom_{\mathscr{C}}(X, Y) \cong Hom_{\mathscr{D}}(H(X), H(Y))$ and $H_{X,Y}$ is a bijection. Hence H is fully faithful as required. This completes the proof. $\square$

Chapter 2

Adjoint pairs of functors

2.1 Definition and examples

Adjoint pairs of functors show up in a variety of different fields in mathematics. Hence, this notion is one of the foundational constructions in basic category theory. With the examples in this section, we will attempt to demonstrate to the reader the ubiquity of adjoint functors in mathematics.

Definition 2.1.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. We say that the pair of functors (F, G) is an **adjoint pair** if the following is satisfied:

If X is an object in $\mathscr{C}$ and Y is an object in $\mathscr{D}$ then there exists a bijection

$$\tau = \tau_{X,Y} : Hom_{\mathscr{D}}(F(X), Y) \to Hom_{\mathscr{C}}(X, G(Y)). \tag{2.1}$$

Furthermore, the bijection is natural in A and B. This means that if $f : A \to A'$ is a morphism in $\mathscr{C}$ and $g : B \to B'$ is a morphism in $\mathscr{D}$ then the following diagram commutes:

$$\begin{array}{ccccc} Hom_{\mathscr{D}}(F(A'), B) & \xrightarrow{F(f)^*} & Hom_{\mathscr{D}}(F(A), B) & \xrightarrow{g_*} & Hom_{\mathscr{D}}(F(A), B') \\ \tau\downarrow & & \downarrow\tau & & \downarrow\tau \\ Hom_{\mathscr{C}}(A', G(B)) & \xrightarrow[f^*]{} & Hom_{\mathscr{C}}(A, G(B)) & \xrightarrow[G(g)_*]{} & Hom_{\mathscr{C}}(A, G(B')) \end{array} \tag{2.2}$$

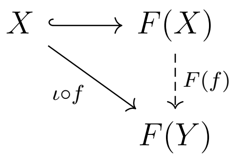
The map f^* denotes precomposition by f whereas g_* denotes composition by g. The LHS square states that the isomorphism $\tau_{A,B}$ is natural in A, whereas the RHS square states that $\tau_{A,B}$ is natural in B.

If (F, G) is an adjoint pair of functors then F is called a **left adjoint functor** and G is called a **right adjoint functor**.

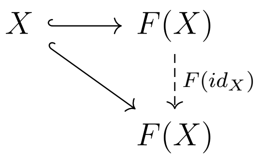
In Definition 2.1.1, the name "adjoint pair" originates from the fact that equation (2.1) looks like the inner product characterisation of the adjoint of a bounded linear operator on a Hilbert space. Our first example of an adjoint pair of functors is one we have already encountered via the construction of a free monoid (see Definition 1.2.5).

Example 2.1.1. Let $U : \mathbf{Mon} \to \mathbf{Set}$ denote the forgetful functor. The functor U is part of an adjoint pair as a left adjoint functor. Let us describe its corresponding right adjoint. Recall from Definition 1.2.5 that if X is a set then $F(X)$ is the free monoid on X. We will turn F into a functor from **Set** to **Mon**.

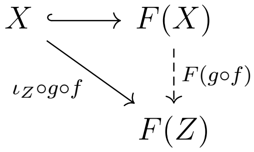
Let X and Y be sets and $f : X \to Y$ be a morphism of sets. In order to make F a functor, we have to construct a monoid morphism from $F(X)$ to $F(Y)$ using f. The key to this is the universal property of the free monoid in Theorem 1.2.4. Let $\iota : Y \hookrightarrow F(Y)$ be the inclusion of sets. Then $\iota \circ f$ is a morphism of sets from X to $F(Y)$ and by the universal property of the free monoid, there exists a unique monoid morphism $F(f) : F(X) \to F(Y)$ such that the following diagram in **Set** commutes:



We will now show that $F : \mathbf{Set} \to \mathbf{Mon}$ is a functor. Let $id_X : X \to X$ denote the identity function on X. Then, $F(id_X)$ is the unique monoid morphism making the following diagram commute:



By uniqueness, $F(id_X) = id_{F(X)}$. Now let Z be another set and $g : Y \to Z$ be a morphism of sets. Then $F(g \circ f) : F(X) \to F(Z)$ is the unique monoid morphism making the following diagram commute:



Here, ι_Z is the inclusion of sets $Z \hookrightarrow F(Z)$. By using Theorem 1.2.4, one can show that $F(g) \circ F(f)$ is another monoid morphism making the above diagram commute. By uniqueness, $F(g) \circ F(f) = F(g \circ f)$ and therefore, $F : \mathbf{Set} \to \mathbf{Mon}$ is a functor. The pair (U, F) is an example of an adjoint pair of functors and a proof of this fact is just careful verification of Definition 2.1.1

Similarly, the forgetful functor $V : \mathbf{Grp} \to \mathbf{Set}$ is part of the adjoint pair (V, F) where $F : \mathbf{Set} \to \mathbf{Grp}$ is the free group functor, which is constructed in a very similar fashion to free monoid functor. See [Lei14, Example 1.2.4 (a)] for a few details on the free group functor.

Example 2.1.2. Our next example of a pair of adjoint functors is from representation theory. Let k be a field and G be a group. Let $H \leq G$ be a subgroup of G. Let $\mathbf{Rep}_G$ be the category whose objects are representations of G (on a k-vector space) and whose morphisms are G-equivariant linear maps. Let (W, ρ_W) be a representation of H. We define the induced representation of W to be the k-vector space

$$Ind_H^G W = \{f : G \to W \mid \text{If } g \in G \text{ and } h \in H \text{ then } f(hg) = \rho_W(h) f(g)\}$$

together with the group morphism

$$\begin{array}{rccc} \pi : & G & \to & Aut(Ind_H^G W) \\ & g & \mapsto & \big(\alpha \mapsto (h \mapsto \alpha(hg))\big). \end{array} \tag{2.3}$$

The point that the construction of induced representations defines a functor

$$\begin{array}{rccc} Ind_H^G : & \mathbf{Rep}_H & \to & \mathbf{Rep}_G \\ & W & \mapsto & Ind_H^G W \\ & \phi : W_1 \to W_2 & \mapsto & (f \mapsto \phi \circ f). \end{array}$$

We also have a restriction functor

$$\begin{array}{rccc} Res_H^G : & \mathbf{Rep}_G & \to & \mathbf{Rep}_H \\ & (V, \rho) & \mapsto & (V, \rho|_H) \\ & \phi : V_1 \to V_2 & \mapsto & \phi \end{array}$$

We claim that the pair (Res_H^G, Ind_H^G) is an adjoint pair of functors. Assume that (V, ρ) is a representation of G over a field k and (W, ν) is a representation of H over k. We define the map $\Phi_{V,W}$ by

$$\begin{array}{rccc} \Phi_{V,W}: & Hom_G(V, Ind_H^G W) & \to & Hom_H(Res_H^G V, W) \\ & \alpha & \mapsto & \big(v \mapsto \alpha(v)(e_G)\big) \end{array}$$

where e_G is the identity element of G. To see that $\Phi_{V,W}$ is well-defined, assume that π is the group morphism in equation (2.3). If $v \in Res_H^G V$ then

$$\begin{aligned} \Phi_{V,W}(\alpha)(\rho(h)v) &= \alpha(\rho(h)v)(e_G) = (\pi(h)\alpha(v))(e_G) \\ &= \alpha(v)(e_G \cdot h) = \alpha(v)(h \cdot e_G) \\ &= \nu(h)\alpha(v)(e_G) \qquad (\text{since } \alpha(v) \in Ind_H^G W) \\ &= \nu(h)\Phi_{V,W}(\alpha)(v). \end{aligned}$$

This demonstrates that $\Phi_{V,W}(\alpha)$ is H-equivariant. Next define the map

$$\begin{array}{rccc} \Psi_{V,W}: & Hom_H(Res_H^G V, W) & \to & Hom_G(V, Ind_H^G W) \\ & \beta & \mapsto & \big(v \mapsto (g \mapsto \beta(\rho(g)v))\big) \end{array}$$

Again, we need to show that $\Psi_{V,W}$ is well-defined. To this end, assume that $\beta \in Hom_H(Res_H^G V, W)$, $v \in V$ and $g \in G$. If $h \in H$ then

$$\begin{aligned} \Psi_{V,W}(\beta)(v)(hg) &= \beta(\rho(hg)v) = \beta(\rho(h)\rho(g)v) \\ &= \nu(h)\beta(\rho(g)v) \qquad (\beta \text{ is } H\text{-equivariant}) \\ &= \nu(h)\Psi_{V,W}(\beta)(v)(g). \end{aligned}$$

So, $\Psi_{V,W}(\beta)(v) \in Ind_H^G W$. Now if $g_2 \in G$ then

$$\begin{aligned} \Psi_{V,W}(\beta)(\rho(g)v)(g_2) &= \beta(\rho(g_2)\rho(g)v) \\ &= \beta(\rho(g_2 g)v) \\ &= \Psi_{V,W}(\beta)(v)(g_2 g) \\ &= \pi(g)\Psi_{V,W}(\beta)(v)(g_2). \end{aligned}$$

The last line follows from the fact that $\Psi_{V,W}(\beta)(v) \in Ind_H^G W$. Hence, $\Psi_{V,W}(\beta)$ must be G-equivariant. We conclude that $\Psi_{V,W}$ is well-defined.

Next, we will show that $\Psi_{V,W}$ and $\Phi_{V,W}$ are inverses of each other. Firstly, assume that $\alpha \in Hom_G(V, Ind_H^G W)$. If $v \in V$ and $g \in G$ then

$$\Psi_{V,W}(\Phi_{V,W}(\alpha))(v)(g) = \Phi_{V,W}(\alpha)(\rho(g)v) = \alpha(\rho(g)v)(e_G)$$
$$= \pi(g)\alpha(v)(e_G) = \alpha(v)(e_G).$$

So, $(\Psi_{V,W} \circ \Phi_{V,W})(\alpha) = \alpha$. Secondly, if $\beta \in Hom_H(Res_H^G V, W)$ then

$$\Phi_{V,W}(\Psi_{V,W}(\beta))(v) = \Psi_{V,W}(\beta)(v)(e_G) = \beta(\rho(e_G)v) = \beta(v).$$

So, $(\Phi_{V,W} \circ \Psi_{V,W})(\beta) = \beta$ and thus, $\Psi_{V,W}$ is a bijective map.

Now we will show that the bijection $\Phi_{V,W}$ is natural as in Definition 2.1.1. To be clear, in Definition 2.1.1, $F = Ind_H^G$ and $G = Res_H^G$. First, assume that $\phi : V \to V'$ is a morphism of representations of G. If $v \in V$ and $\eta \in Hom_G(V', Ind_H^G W)$ then

$$\begin{aligned}\Phi_{V,W}(\eta \circ \phi(v)) &= \Phi_{V,W}(\eta(\phi(v))) \\ &= \eta(\phi(v))(e_G) = \Phi_{V',W}(\eta)(\phi(v)) \\ &= (\Phi_{V',W}(\eta) \circ Res_H^G \phi)(v).\end{aligned}$$

Hence, the RHS square in equation 2.2 commutes. Next, if $\psi : W \to W'$ is a morphism of representations of H, $\delta \in Hom_G(V, Ind_H^G W)$ and $v \in Res_H^G V$ then

$$\begin{aligned}\Phi_{V,W'}(Ind_H^G \psi \circ \delta(v)) &= (Ind_H^G \psi \circ \delta(v))(e_G) \\ &= (\psi \circ \delta(v))(e_G) \\ &= \psi \circ (\delta(v)(e_G)) \\ &= \psi \circ (\Phi_{V,W}(\delta(v))).\end{aligned}$$

So, the entire rectangle in equation 2.2 commutes. We conclude that $\Phi_{V,W}$ is a natural bijection as in Definition 2.1.1 and consequently, the pair (Res_H^G, Ind_H^G) is an adjoint pair of functors. The fact that this pair is an adjoint pair of functors is called *Frobenius reciprocity.*

Example 2.1.3. This example of an adjoint pair of functors is from homological algebra. Assume that A is a commutative ring and M, N, K are A-modules. The functor $Hom_A(N, -)$ is defined by

$$\begin{array}{rccc} Hom_A(N,-): & \textbf{A-Mod} & \to & \textbf{A-Mod} \\ & M & \mapsto & Hom_A(N,M) \\ & f : M \to K & \mapsto & Hom_A(N,f). \end{array}$$

In turn, the A-module morphism $Hom_A(N, f)$ is defined by

$$\begin{array}{rccc} Hom_A(N,f): & Hom_A(N,M) & \to & Hom_A(N,K) \\ & h & \mapsto & f \circ h. \end{array}$$

We define the functor $(-) \otimes_A N$ by

$$\begin{array}{rccc} (-) \otimes_A N: & \textbf{A-Mod} & \to & \textbf{A-Mod} \\ & M & \mapsto & M \otimes_A N \\ & f : M \to K & \mapsto & f \otimes_A N. \end{array}$$

In turn, the A-module morphism $f \otimes_A N$ is defined by

$$\begin{array}{rccc} f \otimes_A N: & M \otimes_A N & \to & K \otimes_A N \\ & m \otimes n & \mapsto & f(m) \otimes n. \end{array}$$

To see that the functors $Hom_A(N, -)$ and $(-) \otimes_A N$ form an adjoint pair, first observe that by direct computation, the map

$$\begin{array}{rccc} \Delta: & Hom_A(M \otimes_A N, K) & \to & Hom_A(M, Hom_A(N,K)) \\ & f & \mapsto & \big(m \mapsto (f \circ \phi)(m, -)\big) \end{array}$$

is bijective where ϕ is the bilinear map

$$\begin{array}{rccc} \phi: & M \times N & \to & M \otimes_A N \\ & (m,n) & \mapsto & m \otimes n \end{array}$$

and the A-module homomorphism $(f \circ \phi)(m, -)$ sends $n \in N$ to $(f \circ \phi)(m, n) = f(m \otimes n)$.

To see that $(-) \otimes_A N$ and $Hom_A(N, -)$ is a pair of adjoint functors, we must show that the rectangle in equation (2.2) commutes with $F = (-) \otimes_A N$ and $G = Hom_A(N, -)$ in Definition 2.1.1. Assume that $f : M \to M'$ is a morphism in **A-Mod**. If $h \in Hom_A(M' \otimes_A N, K)$ then

$$\begin{aligned} \Delta \circ (f \otimes_A N)^*(h) &= m \mapsto h(f(m) \otimes -) \\ &= f^* \circ (m' \mapsto h(m' \otimes -)) \\ &= f^* \circ (m' \mapsto (h \circ \phi)(m', -)) \\ &= (f^* \circ \Delta)(h). \end{aligned}$$

Hence, $f^* \circ \Delta = \Delta \circ (f \otimes_A N)^*$ and the LHS square in equation (2.2) commutes. Next assume that $g : K \to K'$ is a A-module morphism. If $j \in Hom_A(M \otimes_A N, K)$ and $m \in M$ then

$$
\begin{aligned}
(\Delta \circ g_*)(j)(m) &= (g \circ j \circ \phi)(m, -)\\
&= g_*((j \circ \phi)(m, -))\\
&= Hom_A(N, g)_*((j \circ \phi)(m, -))\\
&= (Hom_A(N, g)_* \circ \Delta)(j)(m).
\end{aligned}
$$

So, $\Delta \circ g_* = Hom_A(N, g)_* \circ \Delta$ and $((-) \otimes_A N, Hom_A(N, -))$ is a pair of adjoint functors from **A-Mod** to **A-Mod**. This adjoint pair is known as the *Hom-tensor adjunction.*

In order to gain experience with the definition of an adjoint pair, let us prove the following property regarding adjoint pairs of functors.

Theorem 2.1.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that (F, G) is an adjoint pair.*

1. *If f is an epimorphism in $\mathscr{C}$ then $F(f)$ is an epimorphism in $\mathscr{D}$.*
2. *If g is a monomorphism in $\mathscr{D}$ then $G(g)$ is a monomorphism in $\mathscr{C}$.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories. Assume that (F, G) is a pair of adjoint functors. Since (F, G) is an adjoint pair of functors, if $A \in \mathscr{C}$ and $B \in \mathscr{D}$ then we have a bijection

$$\tau_{A,B} : Hom_{\mathscr{D}}(F(A), B) \to Hom_{\mathscr{C}}(A, G(B))$$

which is natural in both A and B. Recall that this means that the following diagram commutes:

$$
\begin{array}{ccccc}
Hom_{\mathscr{D}}(F(A'), B) & \xrightarrow{F(f)^*} & Hom_{\mathscr{D}}(F(A), B) & \xrightarrow{g_*} & Hom_{\mathscr{D}}(F(A), B')\\
\downarrow{\scriptstyle \tau_{A',B}} & & \downarrow{\scriptstyle \tau_{A,B}} & & \downarrow{\scriptstyle \tau_{A,B'}}\\
Hom_{\mathscr{C}}(A', G(B)) & \xrightarrow[f^*]{} & Hom_{\mathscr{C}}(A, G(B)) & \xrightarrow[G(g)_*]{} & Hom_{\mathscr{C}}(A, G(B'))
\end{array}
$$

where f^* is defined by

$$
\begin{array}{cccc}
f^* : & Hom_{\mathscr{C}}(A', G(B)) & \to & Hom_{\mathscr{C}}(A, G(B))\\
& j & \mapsto & j \circ f
\end{array}
$$

and g_* is defined by

$$\begin{array}{rccc} g_*: & Hom_{\mathscr{C}}(F(A), B) & \to & Hom_{\mathscr{C}}(F(A), B') \\ & k & \mapsto & g \circ k. \end{array}$$

The maps $F(f)^*$ and $G(g)_*$ are defined similarly. Now assume that $f : C \to C'$ is an epimorphism in $\mathscr{C}$ and $g : D \to D'$ is a monomorphism in $\mathscr{D}$.

To show: (a) The morphism $F(f)$ is an epimorphism in $\mathscr{D}$.

(b) The morphism $G(g)$ is a monomorphism in $\mathscr{C}$.

(a) Assume that $u, v : F(C') \to K$ are morphisms in $\mathscr{D}$. Assume that $u \circ F(f) = v \circ F(f)$. So, $F(f)^*(u) = F(f)^*(v)$ and by naturality of τ, the following diagram must commute:

$$\begin{array}{ccc} Hom_{\mathscr{D}}(F(C'), K) & \xrightarrow{F(f)^*} & Hom_{\mathscr{D}}(F(C), K) \\ {\scriptstyle \tau_{C',K}}\downarrow & & \downarrow{\scriptstyle \tau_{C,K}} \\ Hom_{\mathscr{C}}(C', G(K)) & \xrightarrow[f^*]{} & Hom_{\mathscr{C}}(C, G(K)) \end{array}$$

So,

$$\begin{aligned} \tau_{C',K}(v) \circ f &= (f^* \circ \tau_{C',K})(v) \\ &= (\tau_{C,K} \circ F(f)^*)(v) \\ &= (\tau_{C,K} \circ F(f)^*)(u) \\ &= (f^* \circ \tau_{C',K})(u) \\ &= \tau_{C',K}(u) \circ f. \end{aligned}$$

Since f is an epimorphism, $\tau_{C',K}(v) = \tau_{C',K}(u)$. By applying the inverse $\tau^{-1}_{C',K}$ to both sides, we deduce that $u = v$. Therefore, $F(f)$ is an epimorphism in $\mathscr{D}$.

(b) Assume that $w, z : L \to G(D)$ are morphisms in $\mathscr{C}$. Assume that $G(g) \circ w = G(g) \circ z$. Then, $G(g)_*(w) = G(g)_*(z)$. By naturality of τ, the following diagram must commute:

$$\begin{array}{ccc} Hom_{\mathscr{D}}(F(L), D) & \xrightarrow{g_*} & Hom_{\mathscr{D}}(F(L), D') \\ {\scriptstyle \tau_{L,D}}\downarrow & & \downarrow{\scriptstyle \tau_{L,D'}} \\ Hom_{\mathscr{C}}(L, G(D)) & \xrightarrow[G(g)_*]{} & Hom_{\mathscr{C}}(L, G(D')) \end{array}$$

Since $G(g)_*(w) = G(g)_*(z)$ then $(\tau_{L,D'}^{-1} \circ G(g)_*)(w) = (\tau_{L,D'}^{-1} \circ G(g)_*)(z)$ and by commutativity of the above diagram

$$(g_* \circ \tau_{L,D}^{-1})(w) = (\tau_{L,D'}^{-1} \circ G(g)_*)(w) = (\tau_{L,D'}^{-1} \circ G(g)_*)(z) = (g_* \circ \tau_{L,D}^{-1})(z).$$

So, $g \circ \tau_{L,D}^{-1}(w) = g \circ \tau_{L,D}^{-1}(z)$ and since g is a monomorphism, $\tau_{L,D}^{-1}(w) = \tau_{L,D}^{-1}(z)$. By applying the map $\tau_{L,D}$, we obtain $w = z$. Therefore, $G(g)$ is a monomorphism as required. □

2.2 The unit and counit of adjunction

In the next two sections, we will give two different characterisations of adjoint pairs of functors. The first one uses the unit and counit of adjunction. As the name suggests, we will begin by constructing the unit and counit of adjunction from a pair of adjoint functors.

Theorem 2.2.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $L : \mathscr{C} \to \mathscr{D}$ and $R : \mathscr{D} \to \mathscr{C}$ be functors so that (L, R) is an adjoint pair of functors. If $A \in \mathscr{C}$ and $B \in \mathscr{D}$ are objects then we have an isomorphism*

$$\tau_{A,B} : Hom_{\mathscr{D}}(L(A), B) \to Hom_{\mathscr{C}}(A, R(B)).$$

1. *There exists a natural transformation $\eta : id_{\mathscr{C}} \Rightarrow R \circ L$ such that if $f \in Hom_{\mathscr{D}}(L(A), B)$ then $\tau_{A,B}(f) = R(f) \circ \eta_A$.*
2. *There exists a natural transformation $\epsilon : L \circ R \Rightarrow id_{\mathscr{D}}$ such that if $g \in Hom_{\mathscr{C}}(A, R(B))$ then $\tau_{A,B}^{-1}(g) = \epsilon_B \circ L(g)$.*
3. *The composites*

$$L(A) \xrightarrow{L(\eta_A)} L((R \circ L)(A)) = (L \circ R)(L(A)) \xrightarrow{\epsilon_{L(A)}} L(A) \tag{2.4}$$

and

$$R(B) \xrightarrow{\eta_{R(B)}} (R \circ L)(R(B)) = R((L \circ R)(B)) \xrightarrow{R(\epsilon_B)} R(B) \tag{2.5}$$

are the identity morphisms on $L(A)$ and $R(B)$ respectively.

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories. Assume that (L, R) is the adjoint pair of functors defined as above. Assume that if $A \in \mathscr{C}$ and $B \in \mathscr{D}$ are objects then the bijection $\tau_{A,B}$ is defined as above.

If A is an object in $\mathscr{C}$ then define

$$\eta_A = \tau_{A,L(A)}(id_{L(A)}) : A \to (R \circ L)(A).$$

To show: (a) η is a natural transformation from $id_{\mathscr{C}}$ to $R \circ L$.

(a) Assume that $f : A \to A'$ is a morphism in $\mathscr{C}$. We will show that the following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_{A'}} \\ (R \circ L)(A) & \xrightarrow[(R\circ L)(f)]{} & (R \circ L)(A'). \end{array}$$

We compute directly that

$$\begin{aligned} \eta_{A'} \circ f &= \tau_{A',L(A')}(id_{L(A')}) \circ f \\ &= f^* \circ \tau_{A',L(A')}(id_{L(A')}) \\ &= (\tau_{A,L(A')} \circ L(f)^*)(id_{L(A')}) \\ &= (\tau_{A,L(A')})(id_{L(A')} \circ L(f)) \\ &= (\tau_{A,L(A')})(L(f) \circ id_{L(A)}) \\ &= (\tau_{A,L(A')} \circ L(f)_*)(id_{L(A)}) \\ &= R(L(f))_* \circ \tau_{A,L(A)}(id_{L(A)}) \\ &= (R \circ L)(f) \circ \eta_A. \end{aligned}$$

In the above computation, we used the naturality of τ encoded in the following diagrams:

$$\begin{array}{ccc} \mathscr{D}(L(A'), L(A')) & \xrightarrow{L(f)^*} & \mathscr{D}(L(A), L(A')) \\ \downarrow{\scriptstyle \tau_{A',L(A')}} & & \downarrow{\scriptstyle \tau_{A,L(A')}} \\ \mathscr{C}(A', (R \circ L)(A')) & \xrightarrow{f^*} & \mathscr{C}(A, (R \circ L)(A')) \end{array}$$

$$\begin{array}{ccc} \mathscr{D}(L(A), L(A)) & \xrightarrow{L(f)_*} & \mathscr{D}(L(A), L(A')) \\ \downarrow{\scriptstyle \tau_{A,L(A)}} & & \downarrow{\scriptstyle \tau_{A,L(A')}} \\ \mathscr{C}(A, (R \circ L)(A)) & \xrightarrow{R(L(f))_*} & \mathscr{C}(A, (R \circ L)(A')) \end{array}$$

So, η is a natural transformation from $id_{\mathscr{C}}$ to $R \circ L$. Furthermore if $f \in Hom_{\mathscr{D}}(L(A), B)$ then

$$\begin{aligned} \tau_{A,B}(f) &= \tau_{A,B}(f \circ id_{L(A)}) \\ &= (\tau_{A,B} \circ f_*)(id_{L(A)}) \\ &= (R(f)_* \circ \tau_{A,L(A)})(id_{L(A)}) \\ &= R(f)_*(\eta_A) = R(f) \circ \eta_A. \end{aligned}$$

Next if B is an object in $\mathscr{D}$ then define

$$\epsilon_B = \tau^{-1}_{R(B),B}(id_{R(B)}) : (L \circ R)(B) \to B.$$

To show: (b) ϵ is a natural transformation.

(b) Assume that $g : B \to B'$ is a morphism in $\mathscr{D}$. We will show that the following diagram commutes:

$$\begin{array}{ccc} (L \circ R)(B) & \xrightarrow{(L \circ R)(g)} & (L \circ R)(B') \\ \downarrow \epsilon_B & & \downarrow \epsilon_{B'} \\ B & \xrightarrow[g]{} & B' \end{array}$$

Using the naturality of τ, we compute directly that

$$\begin{aligned} g \circ \epsilon_B &= g \circ \tau^{-1}_{R(B),B}(id_{R(B)}) \\ &= (g_* \circ \tau^{-1}_{R(B),B})(id_{R(B)}) \\ &= (\tau^{-1}_{R(B),B'} \circ R(g)_*)(id_{R(B)}) \\ &= \tau^{-1}_{R(B),B'}(R(g) \circ id_{R(B)}) \\ &= \tau^{-1}_{R(B),B'}(id_{R(B')} \circ R(g)) \\ &= (\tau^{-1}_{R(B),B'} \circ R(g)^*)(id_{R(B')}) \\ &= (L(R(g))^* \circ \tau^{-1}_{R(B'),B'})(id_{R(B')}) \\ &= \epsilon_{B'} \circ (L \circ R)(g). \end{aligned}$$

Hence, ϵ is a natural transformation. Furthermore, if $g \in Hom_{\mathscr{C}}(A, R(B))$ then

$$\begin{aligned}\tau^{-1}_{A,B}(g) &= \tau^{-1}_{A,B}(id_{R(B)} \circ g)\\ &= (\tau^{-1}_{A,B} \circ g^*)(id_{R(B)})\\ &= (L(g)^* \circ \tau^{-1}_{R(B),B})(id_{R(B)})\\ &= \epsilon_B \circ L(g).\end{aligned}$$

Finally if A is an object in $\mathscr{C}$ then

$$\epsilon_{L(A)} \circ L(\eta_A) = \tau^{-1}_{A,L(A)}(\eta_A) = (\tau^{-1}_{A,L(A)} \circ \tau_{A,L(A)})(id_{L(A)}) = id_{L(A)}$$

and if B is an object in $\mathscr{D}$ then

$$R(\epsilon_B) \circ \eta_{R(B)} = \tau_{R(B),B}(\epsilon_B) = (\tau_{R(B),B} \circ \tau^{-1}_{R(B),B})(id_{R(B)}) = id_{R(B)}.$$

This completes the proof. □

Definition 2.2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $L : \mathscr{C} \to \mathscr{D}$ and $R : \mathscr{D} \to \mathscr{C}$ be functors so that (L, R) is an adjoint pair of functors. The natural transformation $\eta : id_{\mathscr{C}} \Rightarrow R \circ L$ constructed in Theorem 2.2.1 is called the **unit of adjunction**. The natural transformation $\epsilon : L \circ R \Rightarrow id_{\mathscr{D}}$ constructed in Theorem 2.2.1 is called the **counit of adjunction**

The point of the unit and counit of adjunction is that the existence of natural transformations such that the composites in equations (2.5) and (2.4) are identities is enough to produce an adjoint pair of functors, due to the following theorem.

Theorem 2.2.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $L : \mathscr{C} \to \mathscr{D}$ and $R : \mathscr{D} \to \mathscr{C}$ be functors. Let $\eta : id_{\mathscr{C}} \Rightarrow R \circ L$ and $\epsilon : L \circ R \Rightarrow id_{\mathscr{D}}$ be natural transformations such that if $A \in \mathscr{C}$ and $B \in \mathscr{D}$ are objects then*

$$R(\epsilon_B) \circ \eta_{R(B)} = id_{R(B)} \qquad \textit{and} \qquad \epsilon_{L(A)} \circ L(\eta_A) = id_{L(A)}.$$

Then, (L, R) is an adjoint pair of functors.

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories. Assume that L, R, η and ϵ are defined as above. Assume that $A \in \mathscr{C}$ and $B \in \mathscr{D}$ are categories. We must construct a bijection between $Hom_{\mathscr{D}}(L(A), B)$ and $Hom_{\mathscr{C}}(A, R(B))$.

Assume that $g \in Hom_{\mathscr{D}}(L(A), B)$. Define the morphism $\overline{g} : A \to R(B)$ as the composite

$$A \xrightarrow{\eta_A} (R \circ L)(A) \xrightarrow{R(g)} R(B).$$

Next, define the map

$$\begin{array}{rccc} \tau: & Hom_{\mathscr{D}}(L(A),B) & \to & Hom_{\mathscr{C}}(A,R(B)) \\ & g & \mapsto & \overline{g}. \end{array}$$

Assume that $h \in Hom_{\mathscr{C}}(A, R(B))$. Define $\tilde{h} \in Hom_{\mathscr{D}}(L(A), B)$ to be the composite

$$L(A) \xrightarrow{L(h)} (L \circ R)(B) \xrightarrow{\epsilon_B} B.$$

Analogously to before, we define

$$\begin{array}{rccc} \phi: & Hom_{\mathscr{C}}(A,R(B)) & \to & Hom_{\mathscr{D}}(L(A),B) \\ & h & \mapsto & \tilde{h}. \end{array}$$

We claim that τ and ϕ are inverses of each other. If $g \in Hom_{\mathscr{D}}(L(A), B)$ then

$$\begin{aligned} (\phi \circ \tau)(g) &= \phi(\overline{g}) = \phi(R(g) \circ \eta_A) \\ &= \epsilon_B \circ L(R(g) \circ \eta_A) \\ &= \epsilon_B \circ (L \circ R)(g) \circ L(\eta_A) \\ &= g \circ \epsilon_{L(A)} \circ L(\eta_A) = g \circ id_{L(A)} = g. \end{aligned}$$

Also if $h \in Hom_{\mathscr{C}}(A, R(B))$ then

$$\begin{aligned} (\tau \circ \phi)(h) &= \tau(\tilde{h}) = \tau(\epsilon_B \circ L(h)) \\ &= R(\epsilon_B \circ L(h)) \circ \eta_A \\ &= R(\epsilon_B) \circ (R \circ L)(h) \circ \eta_A \\ &= R(\epsilon_B) \circ \eta_{R(B)} \circ h \\ &= id_{R(B)} \circ h = h. \end{aligned}$$

We conclude that if A is an object in $\mathscr{C}$ and B is an object in $\mathscr{D}$ then τ is a bijection from $Hom_{\mathscr{D}}(L(A), B)$ to $Hom_{\mathscr{C}}(A, R(B))$.

Now, we will show that τ is natural in A and B. Let $f : A \to A'$ be a morphism in $\mathscr{C}$. We have to show that $f^* \circ \tau_{A',B} = \tau_{A,B} \circ L(f)^*$. If $\alpha \in Hom_{\mathscr{D}}(L(A'), B)$ then

$$\begin{aligned}(f^* \circ \tau_{A',B})(\alpha) &= f^* \circ R(\alpha) \circ \eta_{A'} \\ &= R(\alpha) \circ \eta_{A'} \circ f \\ &= R(\alpha) \circ (R \circ L)(f) \circ \eta_A \\ &= R(\alpha \circ L(f)) \circ \eta_A \\ &= R(L(f)^*(\alpha)) \circ \eta_A \\ &= (\tau_{A,B} \circ L(f)^*)(\alpha).\end{aligned}$$

Now let $g : B \to B'$ be a morphism in $\mathscr{D}$. To see that $\tau_{A,B'} \circ g_* = R(g)_* \circ \tau_{A,B}$, we compute directly that if $\beta \in Hom_{\mathscr{D}}(L(A), B)$ then

$$\begin{aligned}(\tau_{A,B'} \circ g_*)(\beta) &= \tau_{A,B'}(g \circ \beta) \\ &= R(g \circ \beta) \circ \eta_A \\ &= R(g) \circ R(\beta) \circ \eta_A \\ &= R(g)_* \circ R(\beta) \circ \eta_A \\ &= (R(g)_* \circ \tau_{A,B})(\beta).\end{aligned}$$

Hence, τ is natural in A and B. So (L, R) is an adjoint pair of functors as required. □

As an application of the unit and counit of adjunction, we will prove the following theorem concerning uniqueness of left and right adjoint functors.

Theorem 2.2.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. If there exists $G : \mathscr{D} \to \mathscr{C}$ such that (F, G) is an adjoint pair of functors then G is the unique right adjoint of F up to unique natural isomorphism. An analogous statement holds for the left adjoint.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories and $F : \mathscr{C} \to \mathscr{D}$ is a functor. Assume that G_1 and G_2 are right adjoints of F.

To show: (a) There exists a natural isomorphism $\eta : G_1 \Rightarrow G_2$.

(b) The natural isomorphism constructed in part (a) is unique.

(a) Assume that A is an object in $\mathscr{C}$ and B is an object in $\mathscr{D}$. If $i \in \{1, 2\}$ then there exist bijections

$$\tau_i : Hom_{\mathscr{D}}(F(A), B) \to Hom_{\mathscr{C}}(A, G_i(B)).$$

The map $\tau_2 \circ \tau_1^{-1}$ is then a bijection from $Hom_{\mathscr{C}}(A, G_1(B))$ to $Hom_{\mathscr{C}}(A, G_2(B))$. To be explicit, we will call this bijection $\psi_{A,B}$. If B is an object in $\mathscr{D}$ then define

$$\eta_B = \psi_{G_1(B),B}(id_{G_1(B)}) \in Hom_{\mathscr{C}}(G_1(B), G_2(B)).$$

To see that the family of morphisms $\{\eta_B\}_{B \in \mathbf{ob}(\mathscr{D})}$ defines a natural transformation $\eta : G_1 \Rightarrow G_2$, assume that $f : X \to X'$ is a morphism in $\mathscr{D}$. By the naturality of τ_1 and τ_2, we have

$$\begin{aligned}
\eta_{X'} \circ G_1(f) &= \psi_{G_1(X'),X'}(id_{G_1(X')}) \circ G_1(f) \\
&= G_1(f)^*((\tau_2 \circ \tau_1^{-1})(id_{G_1(X')})) \\
&= (\tau_2 \circ (F \circ G_1)(f)^*)(\tau_1^{-1}(id_{G_1(X')})) \\
&= (\tau_2 \circ \tau^{-1})(G_1(f)^*(id_{G_1(X')})) \\
&= (\tau_2 \circ \tau_1^{-1})(G_1(f)) \\
&= (\tau_2 \circ \tau_1^{-1})(G_1(f)_*(id_{G_1(X)})) \\
&= \tau_2(f_*(\tau_1^{-1}(id_{G_1(X)}))) \\
&= G_2(f)_*((\tau_2 \circ \tau_1^{-1})(id_{G_1(X)})) \\
&= G_2(f) \circ \psi_{G_1(X),X}(id_{G_1(X)}) \\
&= G_2(f) \circ \eta_X.
\end{aligned}$$

Hence $\eta : G_1 \Rightarrow G_2$ is a natural transformation. To see that η is a natural isomorphism, we interchange the roles of 1 and 2 in the preceding argument. The map $\tau_1 \circ \tau_2^{-1}$ is a bijection from $Hom_{\mathscr{C}}(A, G_2(B))$ to $Hom_{\mathscr{C}}(A, G_1(B))$. To be explicit, we will call this bijection $\varphi_{A,B}$. If B is an object in $\mathscr{D}$ then define

$$\epsilon_B = \varphi_{G_2(B),B}(id_{G_2(B)}) \in Hom_{\mathscr{C}}(G_2(B), G_1(B)).$$

By the same argument as before, $\epsilon : G_2 \Rightarrow G_1$ is a natural transformation.

If $f \in Hom_{\mathscr{C}}(A, G_1(B))$ then

$$\begin{aligned}
\epsilon_B \circ \eta_B \circ f &= \varphi_{G_2(B),B}(id_{G_2(B)}) \circ \psi_{G_1(B),B}(id_{G_1(B)}) \circ f \\
&= (\psi_{G_1(B),B}(id_{G_1(B)}) \circ f)^*(\varphi_{G_2(B),B}(id_{G_2(B)})) \\
&= ((f^* \circ \tau_2 \circ \tau_1^{-1})(id_{G_1(B)}))^*(\varphi_{G_2(B),B}(id_{G_2(B)})) \\
&= ((\tau_2 \circ \tau_1^{-1})(f))^*(\varphi_{G_2(B),B}(id_{G_2(B)})) \\
&= ((\tau_2 \circ \tau_1^{-1})(f))^*((\tau_1 \circ \tau_2^{-1})(id_{G_2(B)})) \\
&= (\tau_1 \circ \tau_2^{-1})((\tau_2 \circ \tau_1^{-1})(f)) \\
&= f.
\end{aligned}$$

Since f is arbitrary, we deduce that $\epsilon_B \circ \eta_B = id_{G_1(B)}$. By a similar computation, $\eta_B \circ \epsilon_B = id_{G_2(B)}$. Therefore $\eta : G_1 \Rightarrow G_2$ is a natural isomorphism as required.

(b) Before we show that $\eta : G_1 \Rightarrow G_2$ is unique, we first observe a particular property about η. If $i \in \{1, 2\}$ then let $\beta_i : id_{\mathscr{C}} \Rightarrow G_i \circ F$ be the unit of adjunction associated to the pair (F, G_i). Let $\gamma_i : F \circ G_i \Rightarrow id_{\mathscr{D}}$ be the counit of adjunction associated to the pair (F, G_i). If A is an object in $\mathscr{C}$ and B is an object in $\mathscr{D}$ then

$$\begin{aligned}
\eta_{F(A)} \circ (\beta_1)_A &= (\tau_2 \circ \tau_1^{-1})(id_{(G_1 \circ F)(A)}) \circ (\beta_1)_A \\
&= \big((\beta_1)_A^* \circ \tau_2 \circ \tau_1^{-1}\big)(id_{(G_1 \circ F)(A)}) \\
&= \big(\tau_2 \circ \tau_1^{-1} \circ (\beta_1)_A^*\big)(id_{(G_1 \circ F)(A)}) \\
&= (\tau_2 \circ \tau_1^{-1})((\beta_1)_A) \\
&= G_2(\tau_1^{-1}((\beta_1)_A)) \circ (\beta_2)_A \\
&= G_2((\gamma_1)_{F(A)} \circ F((\beta_1)_A)) \circ (\beta_2)_A \\
&= G_2(id_{F(A)}) \circ (\beta_2)_A = (\beta_2)_A
\end{aligned}$$

and

$$\begin{aligned}
(\gamma_2)_B \circ F(\eta_B) &= (\gamma_2)_B \circ F((\tau_2 \circ \tau_1^{-1})(id_{G_1(B)})) \\
&= (\tau_2^{-1} \circ \tau_2 \circ \tau_1^{-1})(id_{G_1(B)}) \\
&= \tau_1^{-1}(id_{G_1(B)}) \\
&= (\gamma_1)_B \circ F(id_{G_1(B)}) = (\gamma_1)_B.
\end{aligned}$$

Now since τ_1 and τ_2 are both bijections then the natural transformations $\beta_1, \beta_2, \gamma_1, \gamma_2$ are all natural isomorphisms. So

$$\eta_{F(A)} = (\beta_2)_A \circ (\beta_1)_A^{-1} \qquad \text{and} \qquad F(\eta_B) = (\gamma_2)_B^{-1} \circ (\gamma_1)_B.$$

In particular, $G_2F(\eta_B) = G_2((\gamma_2)_B)^{-1} \circ G_2((\gamma_1)_B)$ and since β_2 is a natural transformation,

$$(\beta_2)_{G_2(B)} \circ \eta_B = G_2F(\eta_B) \circ (\beta_2)_{G_1(B)} = G_2((\gamma_2)_B)^{-1} \circ G_2((\gamma_1)_B) \circ (\beta_2)_{G_1(B)}.$$

Therefore

$$\begin{aligned} G_2((\gamma_1)_B) \circ (\beta_2)_{G_1(B)} &= G_2((\gamma_2)_B) \circ (\beta_2)_{G_2(B)} \circ \eta_B \\ &= id_{G_2(B)} \circ \eta_B = \eta_B. \end{aligned}$$

Hence, the natural transformation $\eta : G_1 \Rightarrow G_2$ is unique because it is created from the units and counits of adjunctions associated to G_1 and G_2. □

2.3 Adjoint pair of functors via initial objects

Definition 2.3.1. Let $\mathscr{C}$ be a category and X be an object in $\mathscr{C}$. We say that X is an **initial object** of $\mathscr{C}$ if the following statement is satisfied: If Y is an object in $\mathscr{C}$ then the set $Hom_{\mathscr{C}}(X, Y)$ has a unique element.

Dually, we say that X is an **terminal object** of $\mathscr{C}$ if the following statement is satisfied: If Y is an object in $\mathscr{C}$ then the set $Hom_{\mathscr{C}}(Y, X)$ has a unique element.

By definition, initial and terminal objects in a category, if they exist, are unique up to a unique isomorphism. In a later section, we will see that the initial object is a special case of a *colimit* and dually, that the terminal object is a special case of a *limit*.

In this section, we will focus on initial objects and how they are used to give another characterisation of an adjoint pair of functors. To do this, we need the following construction of a category from [Bor94a, Section 1.6].

Definition 2.3.2. Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be categories. Let $F : \mathscr{C} \to \mathscr{E}$ and $G : \mathscr{D} \to \mathscr{E}$ be functors. The **comma category** (F, G) is defined in the following manner:

1. The objects of (F, G) are triples (A, f, B) where A is an object in $\mathscr{C}$, B is an object in $\mathscr{D}$ and $f \in Hom_{\mathscr{E}}(F(A), G(B))$.

2. Let (A, f, B) and (A', g, B') be objects in (F, G). A morphism from (A, f, B) to (A', g, B') is a pair (α, β) where $\alpha \in Hom_{\mathscr{C}}(A, A')$, $\beta \in Hom_{\mathscr{D}}(B, B')$ and $g \circ F(\alpha) = G(\beta) \circ f$. That is, the morphism (α, β) in (F, G) makes the following diagram commute:

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\beta)} \\ F(A') & \xrightarrow[g]{} & G(B'). \end{array}$$

Composition in the comma category (F, G) is given by

$$(\alpha', \beta') \circ (\alpha, \beta) = (\alpha' \circ \alpha, \beta' \circ \beta).$$

Definition 2.3.3. Let $\mathscr{C}$ be a category and $\mathbf{1}$ be the category with the single object $*$ and the single morphism id_* (the identity morphism on $*$). Let A be an object in $\mathscr{C}$. Define the functor

$$\begin{array}{rccc} \mathbb{1}_A : & \mathbf{1} & \to & \mathscr{C} \\ & * & \mapsto & A \\ & id_* & \mapsto & id_A. \end{array} \tag{2.6}$$

We will use the functor defined in equation (2.6) to state the next characterisation of an adjoint pair of functors.

Theorem 2.3.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and **1** be the category with the single object $*$. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. The pair (F, G) is an adjoint pair of functors if and only if there exists a natural transformation $\epsilon : id_{\mathscr{C}} \Rightarrow G \circ F$ such that if X is an object in $\mathscr{C}$ then the triple $(*, \epsilon_X, F(X))$ is an initial object in the comma category $(\mathbb{1}_X, G)$.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories and that $\mathbf{1}$ is the category with single object $*$. Assume that $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors.

To show: (a) If (F, G) is an adjoint pair of functors then there exists a natural transformation $\epsilon : id_{\mathscr{C}} \Rightarrow G \circ F$ such that if X is an object in $\mathscr{C}$ then $(*, \epsilon_X, F(X)) \in (\mathbb{1}_X, G)$ is an initial object.

(b) If there exists a natural transformation $\epsilon : id_{\mathscr{C}} \Rightarrow G \circ F$ such that if X is an object in $\mathscr{C}$ then $(*, \epsilon_X, F(X)) \in (\mathbb{1}_X, G)$ is an initial object then (F, G) is an adjoint pair of functors.

(a) Assume that (F, G) is an adjoint pair of functors and X is an object in $\mathscr{C}$. Let $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$ be the unit of adjunction associated to the adjoint pair (F, G). Let $(*, f, B)$ be an object in the comma category $(\mathbb{1}_X, G)$. Then f is a morphism in $\mathscr{C}$ from $\mathbb{1}_X(*) = X$ to $G(B)$. We have a bijection

$$\tau : Hom_{\mathscr{D}}(F(X), B) \to Hom_{\mathscr{C}}(X, G(B)).$$

Let $g \in Hom_{\mathscr{D}}(F(X), B)$ be such that $\tau(g) = f$. By construction of the unit of adjunction in Theorem 2.2.1,

$$f = \tau(g) = G(g) \circ \eta_X.$$

Since τ is a bijection, g is the **unique** morphism making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & (G \circ F)(X) \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle G(g)} \\ X & \xrightarrow[f]{} & G(B). \end{array}$$

In other words, (id_*, g) is the unique morphism in the comma category $(\mathbb{1}_X, G)$ from $(*, \eta_X, F(X))$ to $(*, f, B)$. Since the object $(*, f, B)$ was arbitrary then $(*, \eta_X, F(X))$ is an initial object in $(\mathbb{1}_X, G)$ as required.

(b) Conversely, assume that there exists a natural transformation $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$ such that if X is an object in $\mathscr{C}$ then $(*, \eta_X, F(X)) \in (\mathbb{1}_X, G)$ is an initial object. By Theorem 2.2.2, it suffices to construct a counit of the adjunction for the pair of functors (F, G) and then show that it is unique.

Let Y be an object in $\mathscr{D}$. The object $(*, \eta_{G(Y)}, F(G(Y))) \in (\mathbb{1}_{G(Y)}, G)$ is initial by assumption. So there exists a unique morphism (id_*, ϵ_Y) from $(*, \eta_{G(Y)}, F(G(Y)))$ to $(*, id_{G(Y)}, Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} G(Y) & \xrightarrow{\eta_{G(Y)}} & GFG(Y) \\ {\scriptstyle id_{G(Y)}}\downarrow & & \downarrow{\scriptstyle G(\epsilon_Y)} \\ G(Y) & \xrightarrow[id_{G(Y)}]{} & G(Y) \end{array} \tag{2.7}$$

To see that ϵ is a natural transformation from $F \circ G$ to $id_{\mathscr{D}}$, assume that $f : Y \to Z$ is a morphism in $\mathscr{D}$. Using the commutative square in equation

(2.7) and the naturality of η, we obtain the following commutative diagrams:

$$\begin{array}{ccccc}
G(Y) & \xrightarrow{id_{G(Y)}} & G(Y) & \xrightarrow{id_{G(Y)}} & G(Y) \\
\Big\downarrow{\scriptstyle \eta_{G(Y)}} & & \Big\downarrow{\scriptstyle id_{G(Y)}} & & \Big\downarrow{\scriptstyle G(f)} \\
GFG(Y) & \dashrightarrow_{G(\epsilon_Y)} & G(Y) & \dashrightarrow_{G(f)} & G(Z)
\end{array}$$

$$\begin{array}{ccccc}
G(Y) & \xrightarrow{id_{G(Y)}} & G(Y) & \xrightarrow{id_{G(Y)}} & G(Y) \\
\Big\downarrow{\scriptstyle \eta_{G(Y)}} & & \Big\downarrow{\scriptstyle \eta_{G(Z)} \circ G(f)} & & \Big\downarrow{\scriptstyle G(f)} \\
GFG(Y) & \dashrightarrow_{GFG(f)} & GFG(Z) & \dashrightarrow_{G(\epsilon_Z)} & G(Z)
\end{array}$$

So, we have two morphisms $(id_*, f \circ \epsilon_Y)$ and $(id_*, \epsilon_Z \circ FG(f))$ in $(\mathbb{1}_{G(Y)}, G)$ from $(*, \eta_{G(Y)}, F(G(Y)))$ to $(*, G(f), Z)$ which make the respective diagrams commute. Since the object $(*, \eta_{G(Y)}, F(G(Y)))$ in $(\mathbb{1}_{G(Y)}, G)$ is initial then by uniqueness, $f \circ \epsilon_Y = \epsilon_Z \circ FG(f)$ and thus $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$ is a natural transformation.

By equation (2.7), ϵ and η satisfy the property in equation (2.5). To see that ϵ and η satisfy the property in equation (2.4), assume that X is an object in $\mathscr{C}$. We work with the following diagram in $\mathscr{C}$:

$$\begin{array}{ccccc}
X & \xrightarrow{\eta_X} & GF(X) & \xrightarrow{id_{GF(X)}} & GF(X) \\
\Big\downarrow{\scriptstyle \eta_X} & & \Big\downarrow{\scriptstyle GF(\eta_X)} & & \Big\downarrow{\scriptstyle id_{GF(X)}} \\
GF(X) & \xrightarrow[\eta_{GF(X)}]{} & GFGF(X) & \dashrightarrow_{G(\epsilon_{F(X)})} & GF(X)
\end{array}$$

The LHS square commutes because η is a natural transformation. The bottom side of the rectangle is the identity of $GF(X)$, since equation (2.5) is satisfied by ϵ and η. Consequently, the three paths from the top left X to the bottom right $GF(X)$ are all equal.

We find that $(id_*, id_{F(X)})$ and $(id_*, \epsilon_{F(X)} \circ F(\eta_X))$ are morphisms from $(*, \eta_X, F(X))$ to $(*, \eta_X, F(X))$. Again since $(*, \eta_X, F(X))$ is an initial object in $(\mathbb{1}_X, G)$ then $id_{F(X)} = \epsilon_{F(X)} \circ F(\eta_X)$. Since X is an arbitrary object in $\mathscr{C}$ then equation (2.4) is satisfied by ϵ and η. We conclude that $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$ is a counit of adjunction associated to (F, G).

Finally, we will show that ϵ is the unique counit of adjunction. Assume that $\nu : F \circ G \Rightarrow id_{\mathscr{D}}$ is another counit of adjunction. If Y is an object in $\mathscr{D}$ then we have two morphisms in the comma category $(\mathbb{1}_{G(Y)}, G)$ from $(*, \eta_{G(Y)}, FG(Y))$ to $(*, id_{G(Y)}, Y)$ — (id_*, ϵ_Y) and (id_*, ν_Y). This is because the pairs of natural transformations (η, ϵ) and (η, ν) satisfy equation (2.5). Since $(*, \eta_{G(Y)}, FG(Y))$ is an initial object in $(\mathbb{1}_{G(Y)}, G)$ then $\epsilon_Y = \nu_Y$ and $\epsilon = \nu$. Hence, ϵ is the unique counit of adjunction associated to (F, G) and by Theorem 2.2.2, (F, G) is an adjoint pair of functors as required. □

Example 2.3.1. Let k be a field, k-**Vect** be the category of k-vector spaces and k-**Alg** be the category of algebras over k. We have an adjoint pair of functors (T, F) where $F : k\text{-}\mathbf{Alg} \to k\text{-}\mathbf{Vect}$ is the forgetful functor and $T : k\text{-}\mathbf{Vect} \to k\text{-}\mathbf{Alg}$ is the tensor algebra functor.

Let **1** denote the category with the single object $*$. Let $\epsilon : id_{k\text{-}\mathbf{Vect}} \Rightarrow F \circ T$ be the unit of adjunction associated to (T, F). By Theorem 2.3.1, if V is a k-vector space then the triple $(*, \epsilon_V, T(V))$ is an initial object in the comma category $(\mathbb{1}_V, F)$. This means that if A is a k-algebra and $\overline{f} : V \to F(A)$ is a vector space morphism then there exists a unique k-algebra morphism $f : T(V) \to A$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\epsilon_V} & (F \circ T)(V) \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle F(f)} \\ V & \xrightarrow[\overline{f}]{} & F(A) \end{array}$$

In other words, maps out of tensor algebras are uniquely determined by the images of their generators.

The following result is proved in a similar manner to Theorem 2.3.1. We will use it later.

Theorem 2.3.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Let **1** be the category with a single object $*$.*

1. *F has a left adjoint if and only if the following statement is satisfied: If X is an object in $\mathscr{D}$ then the comma category $(\mathbb{1}_X, F)$ has an initial object.*

2. *F has a right adjoint if and only if the following statement is satisfied: If X is an object in $\mathscr{D}$ then the comma category $(F, \mathbb{1}_X)$ has a terminal object.*

Chapter 3

The Yoneda embedding

3.1 Definition and the Yoneda lemma

Definition 3.1.1. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **locally small** if the following statement is satisfied: If X and Y are objects in $\mathscr{C}$ then the class of morphisms $Hom_{\mathscr{C}}(X,Y)$ is actually a set.

If $\mathscr{C}$ is a locally small category then all of the classes of morphisms are sets. This makes it possible to define the following functor.

Definition 3.1.2. Let $\mathscr{C}$ be a locally small category. Define the **Yoneda embedding** to be the functor

$$\begin{array}{rccc} Y: & \mathscr{C} & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \\ & X & \mapsto & Y(X) = Hom_{\mathscr{C}}(-,X) \\ & f: X \to X' & \mapsto & Y(f) \end{array} \tag{3.1}$$

If $f: X \to X'$ is a morphism in $\mathscr{C}$ then $Y(f)$ is a natural transformation defined by the family of maps

$$\{Y(f)_A : Hom_{\mathscr{C}}(A,X) \to Hom_{\mathscr{C}}(A,X') \mid A \in \mathscr{C}\}$$

where if A is an object in $\mathscr{C}$ then we have the morphism of sets

$$\begin{array}{rccc} Y(f)_A: & Hom_{\mathscr{C}}(A,X) & \to & Hom_{\mathscr{C}}(A,X') \\ & g & \mapsto & f \circ g. \end{array}$$

For two functors $F: \mathscr{C} \to \mathscr{D}$ and $F': \mathscr{C} \to \mathscr{D}$, we write $Nat(F,F')$ to denote the set of natural transformations from F to F'. This notation is adopted from [Mur16]. In Definition 3.1.2, there is a quite a bit of checking to do. One has to check that the Yoneda embedding is a functor, that the

Yoneda embedding maps objects in $\mathscr{C}$ to functors and that it maps morphisms in $\mathscr{C}$ to natural transformations. We will omit all the tedious details here.

The Yoneda lemma is an important result about the Yoneda embedding. We state and prove this below.

Lemma 3.1.1 (Yoneda lemma)**.** *Let $\mathscr{C}$ be a locally small category, $F : \mathscr{C}^{op} \to$ **Set** be a functor and C be an object in $\mathscr{C}$. Define the map*

$$\begin{array}{rccc} \Phi_{C,F}: & Nat(Y(C), F) & \to & F(C) \\ & \alpha & \mapsto & \alpha_C(id_C) \end{array}$$

Explicitly, Y is the functor from equation (3.1), α_C is a morphism of sets from $Y(C)(C) = Hom_{\mathscr{C}}(C, C)$ to $F(C)$ and id_C is the identity map on the object C. Then, $\Phi_{C,F}$ is a bijection, which satisfies the following two properties:

1. If $f : C \to C'$ is a morphism in $\mathscr{C}$ then the following square in ***Set*** *commutes:*

$$\begin{array}{ccc} Nat(Y(C), F) & \xrightarrow{\Phi_{C,F}} & F(C) \\ {\scriptstyle (-)\circ Y(f)}\uparrow & & \uparrow{\scriptstyle F(f)} \\ Nat(Y(C'), F) & \xrightarrow[\Phi_{C',F}]{} & F(C') \end{array} \tag{3.2}$$

2. If $\beta : F \to F'$ is a natural transformation then the following diagram in ***Set*** *commutes:*

$$\begin{array}{ccc} Nat(Y(C), F) & \xrightarrow{\Phi_{C,F}} & F(C) \\ {\scriptstyle \beta\circ(-)}\downarrow & & \downarrow{\scriptstyle \beta_C} \\ Nat(Y(C), F') & \xrightarrow[\Phi_{C,F'}]{} & F'(C) \end{array} \tag{3.3}$$

Proof. Assume that $\mathscr{C}$ is a locally small category and $C \in \mathscr{C}$ is an object. Assume that $F : \mathscr{C}^{op} \to$ **Set** is a functor.

To show: (a) The map $\Phi_{C,F}$ is surjective.

(b) The map $\Phi_{C,F}$ is injective.

(c) Diagram (3.2) commutes.

(d) Diagram (3.3) commutes.

(a) Assume that $X \in F(C)$ and D is an object in $\mathscr{C}^{op}$. Define the map $N(X)_D$ by

$$\begin{array}{rrcl} N(X)_D: & Y(C)(D) = Hom_{\mathscr{C}}(D, C) & \to & F(D) \\ & g & \mapsto & F(g)(X) \end{array} \tag{3.4}$$

Recall that F is a contravariant functor by assumption so that $F(g)$ is a morphism in **Set** from $F(C)$ to $F(D)$.

To show: (aa) $N(X) \in Nat(Y(C), F)$.

(aa) We will show that if $h : D \to D'$ is a morphism in $\mathscr{C}^{op}$ then the following diagram in **Set** commutes:

$$\begin{array}{ccc} Y(C)(D') & \xrightarrow{Y(C)(h)} & Y(C)(D) \\ {\scriptstyle N(X)_{D'}}\downarrow & & \downarrow{\scriptstyle N(X)_D} \\ F(D') & \xrightarrow[F(h)]{} & F(D) \end{array}$$

Assume that $\xi \in Y(C)(D') = Hom_{\mathscr{C}}(D', C)$. We compute directly that

$$\begin{aligned} (N(X)_D \circ Y(C)(h))(\xi) &= (N(X)_D \circ Hom_{\mathscr{C}}(h, C))(\xi) \\ &= N(X)_D(\xi \circ h) \\ &= F(\xi \circ h)(X) \\ &= (F(h) \circ F(\xi))(X) \\ &= (F(h) \circ N(X)_{D'})(\xi). \end{aligned}$$

Hence, the above diagram in **Set** commutes and $N(X) \in Nat(Y(C), F)$.

(a) We claim that $\Phi_{C,F}(N(X)) = X$. Using the definitions of $\Phi_{C,F}$ and $N(X)$, we find that

$$\Phi_{C,F}(N(X)) = N(X)_C(id_C) = F(id_C)(X) = id_{F(C)}(X) = X.$$

Therefore, the map $\Phi_{C,F}$ is surjective.

(b) Assume that $\alpha, \beta \in Nat(Y(C), F)$ such that $\Phi_{C,F}(\alpha) = \Phi_{C,F}(\beta)$. Assume that D is an object in $\mathscr{C}$ and $f \in Hom_{\mathscr{C}}(D, C)$. By naturality of α, the following diagram in **Set** commutes:

$$\begin{array}{ccc} Y(C)(C) & \xrightarrow{Y(C)(f)} & Y(C)(D) \\ {\scriptstyle \alpha_C}\downarrow & & \downarrow{\scriptstyle \alpha_D} \\ F(C) & \xrightarrow[F(f)]{} & F(D) \end{array}$$

We then have

$$\begin{aligned} (F(f) \circ \Phi_{C,F})(\alpha) &= F(f)(\alpha_C(id_C)) \\ &= (\alpha_D \circ Y(C)(f))(id_C) \\ &= \alpha_D(Hom_{\mathscr{C}}(f, C)(id_C)) \\ &= \alpha_D(id_C \circ f) = \alpha_D(f). \end{aligned}$$

Since $\Phi_{C,F}(\alpha) = \Phi_{C,F}(\beta)$ by assumption, $\alpha_C(id_C) = \beta_C(id_C)$. But, β is also a natural transformation between the functors $Y(C)$ and F. So, the following diagram in **Set** commutes:

$$\begin{array}{ccc} Y(C)(C) & \xrightarrow{Y(C)(f)} & Y(C)(D) \\ {\scriptstyle \beta_C}\downarrow & & \downarrow{\scriptstyle \beta_D} \\ F(C) & \xrightarrow{F(f)} & F(D) \end{array}$$

If $f \in Hom_{\mathscr{C}}(D, C) = Y(C)(D)$ then

$$\begin{aligned} \alpha_D(f) &= \alpha_D(id_C \circ f) \\ &= \alpha_D(Hom_{\mathscr{C}}(f, C)(id_C)) \\ &= (\alpha_D \circ Y(C)(f))(id_C) \\ &= F(f)(\alpha_C(id_C)) \\ &= F(f)(\beta_C(id_C)) \quad (\text{since } \alpha_C(id_C) = \beta_C(id_C)) \\ &= (\beta_D \circ Y(C)(f))(id_C) \\ &= \beta_D(Hom_{\mathscr{C}}(f, C)(id_C)) = \beta_D(f). \end{aligned}$$

Therefore, $\alpha_D = \beta_D$. Since the object $D \in \mathscr{C}$ was arbitrary, we deduce that $\alpha = \beta$ as natural transformations from $Y(C)$ to F. Therefore, $\Phi_{C,F}$ is injective.

Combining parts (a) and (b), we deduce that $\Phi_{C,F}$ is indeed a bijective map. Its inverse is given explicitly by

$$\begin{array}{rccc} \Phi_{C,F}^{-1}: & F(C) & \rightarrow & Nat(Y(C),F) \\ & X & \mapsto & N(X) \end{array}$$

where $N(X)$ is the natural transformation in equation (3.4).

(c) Now assume that $f : C \rightarrow C'$ is a morphism in $\mathscr{C}$. We want to show that Diagram (3.2) commutes. Assume that $\alpha \in Nat(Y(C'),F)$. We compute directly that

$$\begin{aligned} (\Phi_{C,F} \circ (-) \circ Y(f))(\alpha) &= \Phi_{C,F}(\alpha \circ Y(f)) \\ &= (\alpha \circ Y(f))_C(id_C) \\ &= (\alpha_C \circ Y(f)_C)(id_C) \\ &= \alpha_C(Y(f)_C(id_C)) \\ &= \alpha_C(f \circ id_C) = \alpha_C(f) \end{aligned}$$

and

$$\begin{aligned} (F(f) \circ \Phi_{C',F})(\alpha) &= F(f)(\alpha_{C'}(id_{C'}) \\ &= (F(f) \circ \alpha_{C'})(id_{C'}) \\ &= (\alpha_C \circ Y(C')(f))(id_{C'}) \quad \text{(Naturality of } \alpha) \\ &= \alpha_C(Hom_{\mathscr{C}}(f,C')(id_{C'})) \\ &= \alpha_C(f). \end{aligned}$$

So, Diagram (3.2) commutes.

(d) Assume that $\beta \in Nat(F,F')$. We want to show that Diagram (3.3) commutes. Assume that $\chi \in Nat(Y(C),F)$. We compute directly that

$$\begin{aligned} (\beta_C \circ \Phi_{C,F})(\chi) &= \beta_C(\chi_C(id_C)) \\ &= (\beta_C \circ \chi_C)(id_C) \\ &= (\beta \circ \chi)_C(id_C) \\ &= \Phi_{C,F'}(\beta \circ \chi) \\ &= (\Phi_{C,F'} \circ \beta \circ (-))(\chi). \end{aligned}$$

Therefore, Diagram (3.3) commutes. This completes the proof. □

In the proof of the Yoneda lemma (Lemma 3.1.1), commutativity of Diagram (3.2) tells us that $\Phi_{C,F}$ is natural with respect to the object $C \in \mathscr{C}$. Correspondingly, commutativity of Diagram (3.3) tells us that $\Phi_{C,F}$ is natural with respect to the functor $F : \mathscr{C}^{op} \to \mathbf{Set}$.

Here is a particularly important consequence of the Yoneda lemma.

Theorem 3.1.2. *Let $\mathscr{C}$ be a locally small category. The Yoneda embedding*

$$\begin{array}{cccc} Y: & \mathscr{C} & \to & \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}) \\ & X & \mapsto & Y(X) = Hom_{\mathscr{C}}(-, X) \\ & f : X \to X' & \mapsto & Y(f) \end{array}$$

defined in Definition 3.1.2 is a fully faithful functor.

Proof. Assume that $\mathscr{C}$ is a locally small category and that Y is the Yoneda embedding. Let X, X' be objects in $\mathscr{C}$. Then, the functor Y induces the mapping

$$Y_{X,X'} : Hom_{\mathscr{C}}(X, X') \to Hom_{\mathcal{F}(\mathscr{C}^{op},\mathbf{Set})}(Y(X), Y(X')) = Nat(Y(X), Y(X')).$$

To show: (a) $Y_{X,X'}$ is bijective.

(a) By Lemma 3.1.1, it suffices to show that $Y_{X,X'}$ is the inverse to the bijection $\Phi_{X,Y(X')} : Nat(Y(X), Y(X')) \to Y(X')(X)$. Assume that $f \in Hom_{\mathscr{C}}(X, X')$. Then,

$$\begin{aligned} (\Phi_{X,Y(X')} \circ Y_{X,X'})(f) &= \Phi_{X,Y(X')}(Y(f)) \\ &= Y(f)_X(id_X) \\ &= f \circ id_X = f. \end{aligned}$$

Hence, $Y_{X,X'}$ is a bijection and the Yoneda embedding is fully faithful as required. □

3.2 Representable functors

In this section, we will apply the Yoneda embedding to provide a characterisation of *representable functors.*

Definition 3.2.1. Let $\mathscr{C}$ be a locally small category and $X : \mathscr{C}^{op} \to \mathbf{Set}$ be a (contravariant) functor. Let Y be the Yoneda embedding in equation (3.1). A **representation** of the functor X is a choice of object $A \in \mathscr{C}$ and an isomorphism from $Y(A)$ to X in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$. We say that X is **representable** if there exists a representation of X.

The Yoneda lemma in Lemma 3.1.1 provides us with another characterisation of a representable **Set**-valued contravariant functor.

Theorem 3.2.1. *Let $\mathscr{C}$ be a locally small category and $X : \mathscr{C}^{op} \to$ **Set** be a functor. Then, X is a representable functor if and only if there exist an object $A \in \mathscr{C}$ and an element $p \in X(A)$ such that if $B \in \mathscr{C}$ and $q \in X(B)$ then there exists a unique morphism $\nu_q : B \to A$ such that $X(\nu_q)(p) = q$.*

Proof. Assume that $\mathscr{C}$ is a locally small category and X is an object in the functor category $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$. Assume that Y is the Yoneda embedding in equation (3.1).

To show: (a) If X is representable then there exist an object $A \in \mathscr{C}$ and an element $p \in X(A)$ such that if $B \in \mathscr{C}$ and $q \in X(B)$ then there exists a unique morphism $\nu_q : B \to A$ such that $X(\nu_q)(p) = q$.

(b) If there exist an object $A \in \mathscr{C}$ and an element $p \in X(A)$ such that if $B \in \mathscr{C}$ and $q \in X(B)$ then there exists a unique morphism $\nu_q : B \to A$ such that $X(\nu_q)(p) = q$. then X is representable.

(a) Assume that X is a representable functor. Then, there exists an object $A \in \mathscr{C}$ and an isomorphism $\alpha : Y(A) \Rightarrow X$. Applying the map $\Phi_{A,X}$ from Lemma 3.1.1, we obtain an element $\Phi_{A,X}(\alpha) = \alpha_A(id_A) \in X(A)$.

Now assume that B is an object in $\mathscr{C}$ and $q \in X(B)$. Since α is an isomorphism in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ then the morphism of sets

$$\alpha_B = Y(A)(B) = Hom_{\mathscr{C}}(B, A) \to X(B)$$

is an isomorphism. Thus, there exists a unique morphism $\rho : B \to A$ such that $\alpha_B(\rho) = q$. So

$$\begin{aligned} X(\rho)(\alpha_A(id_A)) &= X(\rho)(\Phi_{A,X}(\alpha)) \\ &= (X(\rho) \circ \Phi_{A,X})(\alpha) \\ &= \Phi_{B,X}(\alpha \circ Y(\rho)) \qquad \text{(by commutativity of Diagram (3.3))} \\ &= (\alpha \circ Y(\rho))_B(id_B) \\ &= (\alpha_B \circ Y(\rho)_B)(id_B) \\ &= \alpha_B(\rho \circ id_B) = q. \end{aligned}$$

We conclude that the object A in $\mathscr{C}$ and the element $\alpha_A(id_A) \in X(A)$ satisfy the property in the statement of the theorem.

(b) Assume that there exist an object A in $\mathscr{C}$ and an element $p \in X(A)$ which satisfy the property in the statement of the theorem. We need to construct an isomorphism from $Y(A)$ to X. If B is an object in $\mathscr{C}$ then define

$$\begin{array}{rccc} N(X)_B: & Y(A)(B) = Hom_{\mathscr{C}}(B, A) & \to & X(B) \\ & f & \mapsto & X(f)(p). \end{array}$$

This morphism of sets is defined exactly as in equation (3.4). By the proof of Lemma 3.1.1, the family of morphisms $\{N(X)_B\}_{B \in \mathbf{ob}(\mathscr{C})}$ defines a natural transformation $N(X) : Y(A) \Rightarrow X$. To see that it is a natural isomorphism, assume that $q \in X(B)$. By the assumption in the statement of the theorem, there exists a unique morphism $\nu_q : B \to A$ such that

$$N(X)_B(\nu_q) = X(\nu_q)(p) = q.$$

Hence, if B is an object in $\mathscr{C}$ then $N(X)_B$ is an isomorphism of sets and $N(X) : Y(A) \Rightarrow X$ is a natural isomorphism. Therefore the object A in $\mathscr{C}$ and the natural isomorphism $N(X)$ together demonstrate that the functor X is representable. This completes the proof. $\square$

To round this section off, we will apply representable functors to adjoint pairs of functors.

Definition 3.2.2. Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The **formal right adjoint** of F is the functor

$$\begin{array}{rccc} G^{form}: & \mathscr{D} & \to & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \\ & Y & \mapsto & \big(X \mapsto Hom_{\mathscr{D}}(F(X), Y)\big) \\ & f : Y \to Y' & \mapsto & G^{form}(f) : G^{form}(Y) \Rightarrow G^{form}(Y'). \end{array}$$

If Y and Y' are objects in $\mathscr{D}$ and X is an object in $\mathscr{C}$ then the natural transformation $G^{form}(f)$ is defined explicitly by

$$\begin{array}{rccc} G^{form}(f)_X: & Hom_{\mathscr{D}}(F(X), Y) & \to & Hom_{\mathscr{D}}(F(X), Y') \\ & g & \mapsto & f \circ g. \end{array}$$

Before we delve into how representable functors are applied to adjoint pairs of functors, we make the following remark. Let $\mathscr{C}$ be a locally small category. Let $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})^{Rep}$ denote the category of representable functors, which is a subcategory of $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$. By the definition of a representable functor, $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})^{Rep}$ is the essential image of the Yoneda embedding Y on $\mathscr{C}$. By combining Theorem 1.5.1 and Theorem 3.1.2, we deduce that the functor $\mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})^{Rep}$ (the Yoneda embedding with restricted codomain) is an equivalence of categories.

In particular, there exists a functor

$$P : \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})^{Rep} \to \mathscr{C} \tag{3.5}$$

such that the pair (Y, P) is an adjoint pair of functors.

Theorem 3.2.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Let $G^{form} : \mathscr{D} \to \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set})$ be the formal right adjoint of F. Then, F has a right adjoint if and only if the following statement is satisfied: If Y is an object in $\mathscr{D}$ then $G^{form}(Y)$ is a representable functor.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are locally small categories. Assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor and that G^{form} is the formal right adjoint of F.

To show: (a) If F has a right adjoint then if Y is an object in $\mathscr{D}$ then $G^{form}(Y)$ is a representable functor.

(b) If $G^{form}(Y)$ is a representable functor for an object Y in $\mathscr{D}$ then F has a right adjoint.

(a) Assume that F has a right adjoint functor $G : \mathscr{D} \to \mathscr{C}$. If X is an object in $\mathscr{C}$ and Y is an object in $\mathscr{D}$ then we have a bijection

$$\tau_{X,Y} : Hom_{\mathscr{D}}(F(X), Y) \to Hom_{\mathscr{C}}(X, G(Y))$$

which is natural in both X and Y. Let $\mathcal{Y} : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ denote the Yoneda embedding in equation (3.1). Then, the object $G(Y) \in \mathscr{C}$ together with the natural isomorphism of sets

$$\tau^{-1}_{(-),Y} : \mathcal{Y}(G(Y)) \Rightarrow G^{form}(Y)$$

demonstrate that if Y is an object in $\mathscr{D}$ then $G^{form}(Y)$ is a representable functor.

(b) Assume that if Y is an object in $\mathscr{D}$ then $G^{form}(Y)$ is a representable functor. Then $G^{form}(Y)$ is an element of $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})^{Rep}$. Let P be the functor defined in equation (3.5). Define $G = P \circ G^{form}$. By definition, G is a functor from $\mathscr{D}$ to $\mathscr{C}$.

If X is an object in $\mathscr{C}$ then observe that

$$
\begin{aligned}
Hom_{\mathscr{C}}(X, G(Y)) &= Hom_{\mathscr{C}}(X, (P \circ G^{form})(Y)) \\
&\cong Hom_{\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})}(\mathcal{Y}(X), G^{form}(Y)) \\
&\cong G^{form}(Y)(X) \\
&= Hom_{\mathscr{D}}(F(X), Y).
\end{aligned}
$$

In the second line, we used the fact that $(\mathcal{Y}, P)$ is an adjoint pair of functors and in the third line, we used the fact that $G^{form}(Y)$ is a representable functor. The isomorphism in the second line is a natural isomorphism. Hence, we have constructed a natural isomorphism of sets from $Hom_{\mathscr{C}}(X, G(Y))$ to $Hom_{\mathscr{D}}(F(X), Y)$. So, (F, G) is an adjoint pair of functors and G is right adjoint to F as required. $\square$

Chapter 4

Limits and colimits

4.1 Cones and cocones

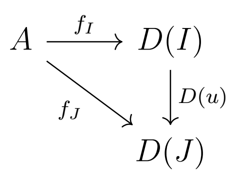
It is likely that the reader is familiar with constructions such as pullbacks, pushouts, products and coproducts in particular categories such as **Grp** and **Top**. In category theory, such constructions are generalised by the notions of limits and colimits. A limit is a cone satisfying a universal property and similarly, a colimit is a cocone satisfying a universal property.

Definition 4.1.1. Let $\mathscr{C}$ be a category and $\mathbf{I}$ be a small category. A functor $\mathbf{I} \to \mathscr{C}$ is called a **diagram** in $\mathscr{C}$ of shape $\mathbf{I}$.

Definition 4.1.2. Let $\mathscr{C}$ be a category, $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. A **cone** on D is an object $A \in \mathscr{C}$, called the **vertex** of the cone, together with a family

$$\big(f_I : A \to D(I)\big)_{I \in \mathbf{I}}$$

of morphisms in $\mathscr{C}$ such that if $u : I \to J$ is a morphism in $\mathbf{I}$ then the triangle in $\mathscr{C}$ below commutes:



Dually, we also have the definition of a cocone.

Definition 4.1.3. Let $\mathscr{C}$ be a category and $\mathbf{I}$ be a small category. Let $D : \mathbf{I} \to \mathscr{C}$ be a functor. A **cocone** on D is an object $A \in \mathscr{C}$, together with a collection of morphisms

$$\big(f_I : D(I) \to A\big)_{I \in \mathbf{I}}$$

such that if $u : I \to J$ is a morphism in $\mathbf{I}$ then the following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} A & \xleftarrow{\ f_I\ } & D(I) \\ & {\scriptstyle f_J}\nwarrow & \big\downarrow {\scriptstyle D(u)} \\ & & D(J) \end{array}$$

Example 4.1.1. Let $\mathscr{C}$ be a category and $\mathbf{P}$ be the small category depicted pictorially by

$$\begin{array}{ccc} & & \bullet \\ & & \big\downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Let $D : \mathbf{P} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{P}$ to

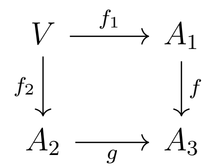
$$\begin{array}{ccc} & & A_1 \\ & & \big\downarrow {\scriptstyle f} \\ A_2 & \xrightarrow[g]{} & A_3 \end{array}$$

A cone on $D : \mathbf{P} \to \mathscr{C}$ is an object $V \in \mathscr{C}$ (the vertex), together with a family of morphisms $f_1 : V \to A_1$, $f_2 : V \to A_2$ and $f_3 : V \to A_3$ such that the following triangles in $\mathscr{C}$ commute:

$$\begin{array}{ccc} V & \xrightarrow{\ f_1\ } & A_1 \\ & {\scriptstyle f_3}\searrow & \big\downarrow {\scriptstyle f} \\ & & A_3 \end{array}$$

$$\begin{array}{ccc} V & & \\ {\scriptstyle f_2}\big\downarrow & \searrow{\scriptstyle f_3} & \\ A_2 & \xrightarrow[g]{} & A_3 \end{array}$$

We can combine these two commutative triangles to find that the cone of D is the commutative square in $\mathscr{C}$

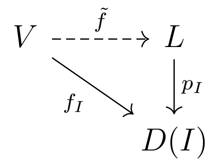


Below, we define limits and colimits.

Definition 4.1.4. Let $\mathscr{C}$ be a category, $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. A **limit** of D is a cone

$$\big(p_I : L \to D(I)\big)_{I\in\mathbf{I}}$$

satisfying the following universal property: If we have another cone $\big(f_I : V \to D(I)\big)_{I\in\mathbf{I}}$ on D then there exists a unique morphism $\tilde{f} : V \to L$ such that if $I \in \mathbf{I}$ then the following diagram in $\mathscr{C}$ commutes:



In a common abuse of notation, we refer to L as the limit of D. We write the limit L as $\varprojlim_{\mathbf{I}} D$.

Let $D : \mathbf{I} \to \mathscr{C}$ be a diagram in a category $\mathscr{C}$. The universal property associated with a limit L of D can be interpreted as the bijective correspondence

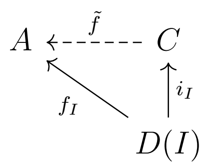
$$\begin{array}{ccc} \{\text{Morphisms } A \to L\} & \leftrightarrow & \{\text{Cones on } D \text{ with vertex } A\} \\ g : A \to L & \mapsto & \big(p_I \circ g : A \to D(I)\big)_{I\in\mathbf{I}} \\ \tilde{f} : A \to L & \leftarrow\!\shortmid & \big(f_I : A \to D(I)\big)_{I\in\mathbf{I}} \end{array} \tag{4.1}$$

The maps $p_I : L \to D(I)$ are the morphisms accompanying the limit L. The universal property of the limit provides the direction from "right to left" in the above correspondence — from a cone on D with vertex A to a unique morphism from A to L.

Definition 4.1.5. Let $\mathscr{C}$ be a category, $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. A **colimit** of D is a cocone

$$\big(i_I : D(I) \to C\big)_{I\in\mathbf{I}}$$

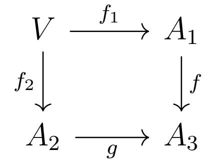
which satisfies the following universal property: If $\big(f_I : D(I) \to A\big)_{I \in \mathbf{I}}$ is another cone of D then there exists a unique morphism $\tilde{f} : C \to A$ such that if $I \in \mathbf{I}$ then the following diagram in $\mathscr{C}$ commutes:



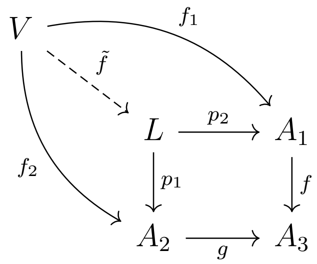
4.2 Examples of limits and colimits in an arbitrary category

This section is dedicated to well-known examples of limits and colimits.

Example 4.2.1. Let us return to Example 4.1.1. The cone of D in the example is the following commutative square in $\mathscr{C}$:



A limit of the diagram $D : \mathbf{P} \to \mathscr{C}$ in $\mathscr{C}$ is another cone, which consists of an object $L \in \mathscr{C}$ and morphisms $p_j : L \to A_j$ for $j \in \{1, 2, 3\}$. By its universal property, there exists a unique morphism $\tilde{f} : V \to L$ such that if $j \in \{1, 2, 3\}$ then $p_j \circ \tilde{f} = f_j$. This is equivalent to saying that the following diagram in $\mathscr{C}$ commutes:



Note that the equation $p_3 \circ \tilde{f} = f_3$ is extraneous data and can be deduced from the commutativity of the above diagram. Indeed, we have

$$f_3 = f \circ f_1 = f \circ (p_2 \circ \tilde{f}) = (f \circ p_2) \circ \tilde{f} = p_3 \circ \tilde{f}.$$

The specific limit constructed in Example 4.2.1 is a well-known construction which merits the following definition.

Definition 4.2.1. Let $\mathscr{C}$ be a category. Suppose that we have the following diagram in $\mathscr{C}$:

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[s]{} & Z \end{array}$$

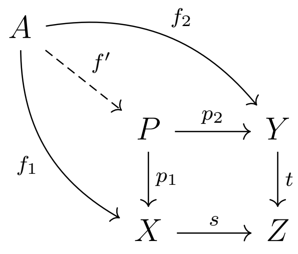
A **pullback** of the above diagram is an object P of $\mathscr{C}$, together with morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ such that firstly, the square below commutes:

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[s]{} & Z \end{array}$$

Secondly, the pullback satisfies the following universal property: If we have a commutative square in $\mathscr{C}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[s]{} & Z \end{array}$$

then there exists a unique morphism $f' : A \to P$ such that the two triangles in the below diagram commute:



Here is our next specific example of a limit.

Example 4.2.2. Let $\mathscr{C}$ be a category and $\mathbf{E}$ be the small category depicted pictorially by

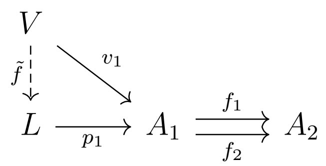
$$\bullet \rightrightarrows \bullet$$

Let $D : \mathbf{E} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{E}$ to

$$A_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} A_2$$

A cone on $D : \mathbf{E} \to \mathscr{C}$ is an object $V \in \mathscr{C}$, together with morphisms $v_1 : V \to A_1$ and $v_2 : V \to A_2$ such that $f_1 \circ v_1 = v_2$ and $f_2 \circ v_1 = v_2$. Hence, a cone on D is a morphism $v_1 : V \to A_1$ satisfying $f_1 \circ v_1 = f_2 \circ v_1$.

A limit of the diagram $D : \mathbf{E} \to \mathscr{C}$ is another cone $L \in \mathscr{C}$ with accompanying morphisms $p_1 : L \to A_1$ and $p_2 : L \to A_2$ such that there exists a unique morphism $\tilde{f} : V \to L$ making the following diagram in $\mathscr{C}$ commute:



Similarly to the construction of a pullback in Example 4.2.1, the equation $p_2 \circ \tilde{f} = v_2$ is an extraneous condition, which can be determined from the commutative diagram above. We have

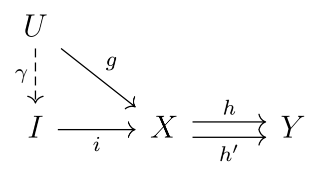
$$v_2 = f_2 \circ v_1 = f_2 \circ (p_1 \circ \tilde{f}) = (f_2 \circ p_1) \circ \tilde{f} = p_2 \circ \tilde{f}.$$

In Example 4.2.2, we have successfully constructed another specific example of limit — the equalizer.

Definition 4.2.2. Let $\mathscr{C}$ be a category and consider the following diagram in $\mathscr{C}$:

$$I \xrightarrow{i} X \underset{h'}{\overset{h}{\rightrightarrows}} Y$$

We say that the morphism $i : I \to X$ is an **equalizer** of the pair (h, h') if firstly, $h \circ i = h' \circ i$ and secondly, i satisfies the following universal property: If $g : U \to X$ is a morphism satisfying $h \circ g = h' \circ g$ then there exists a unique morphism $\gamma : U \to I$ such that the the triangle in the below diagram commutes:



The equalizer i of (h, h') is often denoted by $eq(h, h')$.

Our next example of a limit is likely the most recognisable.

Example 4.2.3. Let $\mathscr{C}$ be a category and $\mathbf{T}$ be the small category depicted pictorially by

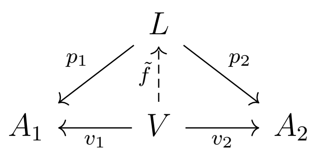
$$\bullet \qquad \bullet$$

Let $D : \mathbf{T} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{T}$ to

$$A_1 \qquad A_2$$

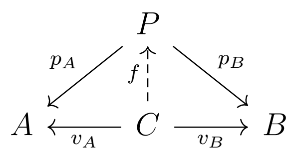
A cone on $D : \mathbf{T} \to \mathscr{C}$ is an object $V \in \mathscr{C}$, together with morphisms $v_1 : V \to A_1$ and $v_2 : V \to A_2$. There are no commutative diagrams to deal with here because there are no morphisms in $\mathbf{T}$.

A limit of the diagram $D : \mathbf{T} \to \mathscr{C}$ is another cone $L \in \mathscr{C}$ with accompanying morphisms $p_1 : L \to A_1$ and $p_2 : L \to A_2$ such that there exists a unique morphism $\tilde{f} : V \to L$ making the following diagram in $\mathscr{C}$ commute:



Similarly to pullbacks and equalizers, Example 4.2.3 leads us to the next definition.

Definition 4.2.3. Let $\mathscr{C}$ be a category and A, B be objects in $\mathscr{C}$. The **product** of A and B is a triple (P, p_A, p_B) consisting of an object P in $\mathscr{C}$ and two morphisms $p_A : P \to A$ and $p_B : P \to B$. Furthermore, the triple satisfies the following universal property: If C is an object in $\mathscr{C}$ and $v_A : C \to A$ and $v_B : C \to B$ are morphisms then there exists a unique morphism $f : C \to P$ such that the following diagram in $\mathscr{C}$ commutes:



We remark that if $\mathscr{C}$ has products then it has **all finite products**. If $n \in \mathbb{Z}_{>0}$ then a finite product is a limit of a diagram from the category with

n objects and no non-identity morphisms. This assertion is straightforward to check by the universal property of a limit.

Now in the following example, we will show that a terminal object can also be thought of as a specific example of a limit.

Example 4.2.4. Let $\mathscr{C}$ be a category and $\mathbf{0}$ be the empty category, with no morphisms or objects. Let $D : \mathbf{0} \to \mathscr{C}$ denote the diagram in $\mathscr{C}$ which sends $\mathbf{0}$ to the empty subcategory of $\mathscr{C}$.

A cone on $D : \mathbf{0} \to \mathscr{C}$ is just an object $V \in \mathscr{C}$. This time, there are no accompanying morphisms because there are no objects in the empty subcategory $D(\mathbf{0})$ of $\mathscr{C}$.

A limit of the diagram $D : \mathbf{0} \to \mathscr{C}$ is another object $L \in \mathscr{C}$ such that there exists a unique morphism $t : V \to L$. Therefore, the limit L of the diagram D is a terminal object in $\mathscr{C}$.

By similar arguments to the specific limits constructed in Examples 4.2.1, 4.2.2, 4.2.3 and 4.2.4, we also obtain important examples of colimits which directly parallel the limits we know so far. For instance, the notion of an initial object is a colimit of a diagram stemming from the empty category $\mathbf{0}$.

Since said arguments are similar, we will end this section by defining the analogous colimits to pullbacks, equalizers and products.

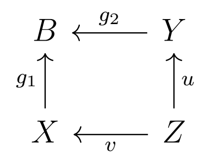
Definition 4.2.4. Let $\mathscr{C}$ be a category. Suppose that we have the following diagram in $\mathscr{C}$:

$$\begin{array}{ccc} & & Y \\ & & \uparrow{\scriptstyle u} \\ X & \xleftarrow[v]{} & Z \end{array}$$

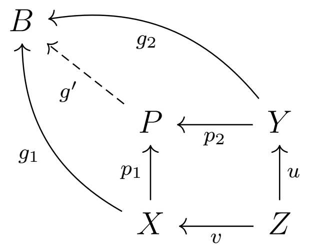
A **pushout** of the above diagram is an object P of $\mathscr{C}$, together with morphisms $p_1 : X \to P$ and $p_2 : Y \to P$ such that firstly, the square below commutes:

$$\begin{array}{ccc} P & \xleftarrow{p_2} & Y \\ {\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle u} \\ X & \xleftarrow[v]{} & Z \end{array}$$

Secondly, the pushout must satisfy the following universal property: If we have a commutative square in $\mathscr{C}$ of the form



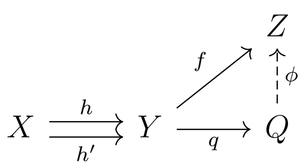
there exists a unique morphism $g' : P \to B$ such that the two triangles in the below diagram commute:



Definition 4.2.5. Let $\mathscr{C}$ be a category and consider the following diagram in $\mathscr{C}$:

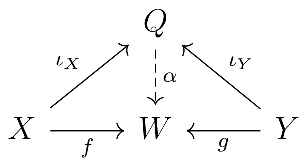
$$X \underset{h'}{\overset{h}{\rightrightarrows}} Y \xrightarrow{q} Q$$

We say that the morphism $q : Y \to Q$ is a **coequalizer** of the pair (h, h') if firstly, $q \circ h = q \circ h'$ and secondly, q satisfies the following universal property: If $f : Y \to Z$ is a morphism in $\mathscr{C}$ satisfying $f \circ h = f \circ h'$ then there exists a unique morphism $\phi : Q \to Z$ such that the the triangle in the below diagram commutes:



The coequalizer q of (h, h') is often denoted by $coeq(h, h')$.

Definition 4.2.6. Let $\mathscr{C}$ be a category and X, Y be objects in $\mathscr{C}$. The **coproduct** of X and Y is a triple (Q, ι_X, ι_Y) consisting of an object Q and two morphisms $\iota_X : X \to Q$ and $\iota_Y : Y \to Q$. Furthermore, the triple satisfies the following universal property: If we have two morphisms $f : X \to W$ and $g : Y \to W$ in $\mathscr{C}$ then there exists a unique morphism $\alpha : Q \to W$ such that the following diagram commutes:



4.3 Examples of limits and colimits in particular categories

How does one show that pullbacks exist in the category **Set**? According to the definition of a pullback, one would construct a set and two functions and then show that the resulting triple satisfies the universal property of the pullback. In the previous section, we gave well-known examples of limits and colimits in an arbitrary category. To complement this, we will now provide examples of limits and colimits in well-known categories.

Example 4.3.1. To begin, we will show that equalizers exist in **Set**. Suppose that we have the following diagram in **Set**:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

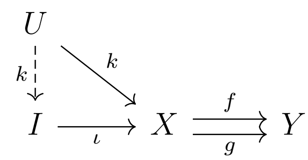
Define the set I by

$$I = \{x \in X \mid f(x) = g(x)\}.$$

We claim that the inclusion map $\iota : I \hookrightarrow X$ is the equalizer of the pair of functions (f, g). By definition of I, if $x \in I$ then

$$f(\iota(x)) = f(x) = g(x) = g(\iota(x))$$

and $f \circ \iota = g \circ \iota$. Now suppose that we have a morphism of sets $k : U \to X$ satisfying $f \circ k = g \circ k$. Then, im $k \subseteq I$. So, the morphism $k : U \to I$, which is k with codomain restricted to I makes the following diagram in **Set** commute:



To see that $k : U \to I$ is unique, suppose that $\ell : U \to I$ is another morphism such that $k = \iota \circ \ell$. If $u \in U$ then

$$k(u) = (\iota \circ \ell)(u) = \ell(u)$$

and $k = \ell$. Hence, $\iota : I \to X$ is the equalizer of the pair (f, g) in **Set**.

Example 4.3.2. In this example, we will show that coproducts exist in the category **Grp**. Assume that G and H are groups. The **free product** of G and H, denoted by $G \star H$, is the set of all reduced words of the form

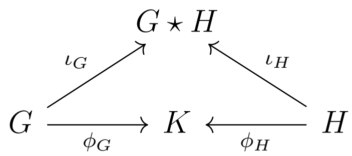
$$g_1 h_1 g_2 h_2 \dots g_k h_k$$

where $g_1, \dots, g_k \in G$ and $h_1, \dots, h_k \in H$. The group operation on $G \star H$ is the concatenation of words, followed by reduction.

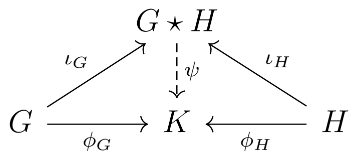
We will describe how to reduce a word. Let e_G and e_H be the identity elements of G and H respectively. Suppose that $g_1 h_1 \dots g_k h_k \in G \star H$. If there exists $i \in \{1, 2, \dots k\}$ such that $g_i = e_G$ or $h_i = e_H$ then we remove e_G or e_H from the word. If there is an instance of $g_j g_{j+1}$ or $h_i h_{i+1}$ then we reduce the word by considering the product $g_j g_{j+1}$ as one element of G (rather than two) or the product $h_i h_{i+1}$ as one element of H (rather than two).

The words $e_G, e_H \in G \star H$ are the empty words (words of length zero). This is the identity element $e_{G \star H}$ of $G \star H$.

Now define the inclusion maps $\iota_G : G \to G \star H$ and $\iota_H : H \to G \star H$ by $\iota_G(g) = g$ and $\iota_H(h) = h$. Suppose that we have the following diagram in **Grp**:



We want to construct a unique morphism $\psi : G \star H \to K$ such that the following diagram commutes:



Define the map ψ by

$$\begin{array}{rccc}\psi: & G\star H & \to & K\\ & g_1h_1\dots g_kh_k & \mapsto & \phi_G(g_1)\phi_H(h_1)\dots\phi_G(g_k)\phi_H(h_k)\end{array}$$

As a preliminary observation, we have

$$\psi(e_{G\star H}) = \phi_G(e_G) = \phi_H(e_H) = e_K.$$

To see that ψ is a group morphism, assume that $g_1h_1\dots g_kh_k$ and $g_1'h_1'\dots g_l'h_l'$ are two reduced words in $G\star H$. If the concatenation $g_1h_1\dots g_l'h_l'$ is already a reduced word then

$$\begin{aligned}\psi(g_1h_1\dots g_kh_kg_1'h_1'\dots g_l'h_l') &= \phi_G(g_1)\phi_H(h_1)\dots\phi_G(g_l')\phi_H(h_l')\\ &= (\phi_G(g_1)\phi_H(h_1)\dots\phi_G(g_k)\phi_H(h_k))\\ &\quad\ (\phi_G(g_1')\phi_H(h_1')\dots\phi_G(g_l')\phi_H(h_l'))\\ &= \psi(g_1h_1\dots g_kh_k)\psi(g_1'h_1'\dots g_l'h_l').\end{aligned}$$

If the concatenation $g_1h_1\dots g_l'h_l'$ is not a reduced word then there are two cases which can occur.

Case 1: If e_G or e_H appears in our word, we remove it by the reduction process. Since $\phi_G(e_G) = \phi_H(e_H) = e_K$, any terms of the form $\phi_G(e_G)$ and $\phi_H(e_H)$ in $\psi(g_1h_1\dots g_l'h_l')$ are removed from the product.

Case 2: If g_ig_{i+1} or h_ih_{i+1} appears in our word, we consider them as a single element of G and H respectively in the word. Since ϕ_G and ϕ_H are group morphisms then $\phi_G(g_ig_{i+1}) = \phi_G(g_i)\phi_G(g_{i+1})$ and $\phi_H(h_ih_{i+1}) = \phi_H(h_i)\phi_H(h_{i+1})$. So we simply rewrite the product $\psi(g_1h_1\dots g_l'h_l')$ by replacing $\phi_G(g_i)\phi_G(g_{i+1})$ with $\phi_G(g_ig_{i+1})$ and similarly for $\phi_H(h_i)\phi_H(h_{i+1})$.

These two cases show that ψ respects the reduction process in $G\star H$. Hence,

$$\psi(g_1h_1\dots g_kh_kg_1'h_1'\dots g_l'h_l') = \psi(g_1h_1\dots g_kh_k)\psi(g_1'h_1'\dots g_l'h_l')$$

even if the concatenation $g_1h_1\dots g_kh_kg_1'h_1'\dots g_l'h_l'$ is not a reduced word. So ψ is a group morphism which satisfies by direct computation, $\psi\circ\iota_G = \phi_G$ and $\psi\circ\iota_H = \phi_H$.

Finally to see that ψ is a unique group morphism, assume that $\psi' : G\star H \to K$ is another group morphism such that $\psi'\circ\iota_G = \phi_G$ and

$\psi' \circ \iota_H = \phi_H$. If $g \in G$ and $h \in H$ then $\psi'(g) = \phi_G(g) = \psi(g)$ and $\psi'(h) = \phi_H(h) = \psi(h)$. Since ψ' and ψ are group morphisms then $\psi' = \psi$ on all of $G \star H$. Hence, ψ must be unique and the triple $(G \star H, \iota_G, \iota_H)$ defines a coproduct in **Grp**.

Free products of groups and free products with amalgamation (which are pushouts in **Grp**) feature prominently in the Seifert-Van Kampen theorem, a useful tool for computing the fundamental group of a wide variety of topological spaces. See [Bre93, Chapter III, Section 9] for details on the Seifert-Van Kampen theorem.

Example 4.3.3. In this example, we will construct pushouts in the category **Top**. Suppose we have the following diagram in **Top**:

$$\begin{array}{ccc} & & Y \\ & & \uparrow{\scriptstyle u} \\ X & \xleftarrow[v]{} & Z \end{array}$$

Consider the disjoint union $X \sqcup Y$ of the topological spaces X and Y. We define an equivalence relation on $X \sqcup Y$ by saying that if $x \in X$ and $y \in Y$ then $x \sim y$ if and only if there exists a $z \in Z$ such that $v(z) = x$ and $u(z) = y$. In other words, $\sim$ is the smallest equivalence relation generated by pairs of the form $(v(z), u(z))$ where $z \in Z$.

Next, we define $X \sqcup_Z Y$ to be the quotient topological space $(X \sqcup Y)/\sim$. We have continuous functions $\iota_X : X \to X \sqcup_Z Y$ and $\iota_Y : Y \to X \sqcup_Z Y$ defined by $\iota_X(x) = [x]$ and $\iota_Y(y) = [y]$. From the definition of $X \sqcup_Z Y$, it is straightforward to verify that the following diagram commutes:

$$\begin{array}{ccc} X \sqcup_Z Y & \xleftarrow{\iota_Y} & Y \\ {\scriptstyle \iota_X}\uparrow & & \uparrow{\scriptstyle u} \\ X & \xleftarrow[v]{} & Z \end{array}$$

To see that the universal property of the pushout is satisfied, suppose that we have the following commutative square in **Top**:

$$\begin{array}{ccc} W & \xleftarrow{g_2} & Y \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle u} \\ X & \xleftarrow[v]{} & Z \end{array}$$

Define the map $\beta : X \sqcup_Z Y \to W$ by

$$\begin{array}{rccc} \beta: & X \sqcup_Z Y & \to & W \\ & [x] & \mapsto & g_1(x) \\ & [y] & \mapsto & g_2(y). \end{array}$$

How do we know that β is a well-defined continuous map? By the universal property of the quotient in **Top**, it suffices to construct a continuous map $\beta' : X \sqcup Y \to W$ such that if $z \in Z$ then $\beta'(u(z)) = \beta'(v(z))$. We will make use of the homeomorphism

$$Cts(X \sqcup Y, W) \cong Cts(X, W) \times Cts(Y, W) \tag{4.2}$$

where $Cts(X, W)$ is the space of continuous functions from X to W. Now $g_1 \in Cts(X, W)$ and $g_2 \in Cts(Y, W)$. By the homeomorphism in equation (4.2), the pair (g_1, g_2) induces the continuous function $\beta' : X \sqcup Y \to W$ which has the desired property because $g_1 \circ v = g_2 \circ u$. Thus, $\beta : X \sqcup_Z Y \to W$ is a well-defined continuous function.

If $x \in X$ and $y \in Y$ then

$$\beta(\iota_X(x)) = \beta([x]) = g_1(x) \qquad \text{and} \qquad \beta(\iota_Y(y)) = \beta([y]) = g_2(y).$$

It remains to show uniqueness. Suppose that $\beta^* : X \sqcup_Z Y \to W$ is another continuous function which satisfies $\beta^* \circ \iota_X = g_1$ and $\beta^* \circ \iota_Y = g_2$. If $x \in X$ and $y \in Y$ then

$$\beta([x]) = \beta^*([x]) \qquad \text{and} \qquad \beta([y]) = \beta^*([y]).$$

So $\beta = \beta^*$ and therefore, the triple $(X \sqcup_Z Y, \iota_X, \iota_Y)$ is a pushout in **Top**.

Pushouts in **Top** play an important role in the construction of *finite CW-complexes*, which are fundamental objects of study in algebraic topology. See [Mur21] for a brief discussion of this.

4.4 Finitely (co)complete categories

Many of the categories we know, such as **Set**, **Grp** and **Top**, are examples of finitely complete and finitely cocomplete categories; the main subject of this section.

Definition 4.4.1. Let $\mathscr{C}$ be a category, $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. We say that a (co)limit of D is **finite** if the small category $\mathbf{I}$ is a finite category — $\mathbf{I}$ has finitely many objects and morphisms.

Definition 4.4.2. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **finitely complete** if it has all finite limits. Similarly, we say that $\mathscr{C}$ is **finitely cocomplete** if it has all finite colimits.

By definition, it seems impossible to check that a given category is finitely complete or finitely cocomplete. Fortunately, there are equivalent characterisations of finitely complete/cocomplete categories which rectify this issue.

Theorem 4.4.1. *Let $\mathscr{C}$ be a category. The following are equivalent:*

1. *$\mathscr{C}$ is finitely complete,*
2. *$\mathscr{C}$ has pullbacks and a terminal object,*
3. *$\mathscr{C}$ has equalizers and products.*

Proof. Assume that $\mathscr{C}$ is a category. If $\mathscr{C}$ is finitely complete then it has finite limits and hence, pullbacks and a terminal object. Next, assume that $\mathscr{C}$ has pullbacks and a terminal object. By using pullbacks and the terminal object, we will construct equalizers and products in $\mathscr{C}$.

Assume that $h, h' : X \to Y$ is a pair of morphisms in $\mathscr{C}$. Since $\mathscr{C}$ has products, we can consider the following diagram in $\mathscr{C}$:

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle (h,\,h')} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

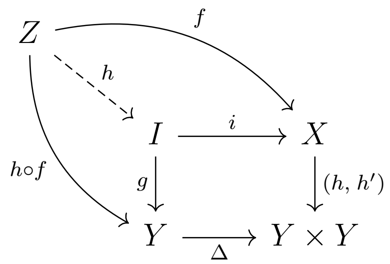
where $\Delta = (id_Y, id_Y)$ is the diagonal map. Since $\mathscr{C}$ has pullbacks, we can form the pullback square of the above diagram:

$$\begin{array}{ccc} I & \xrightarrow{i} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle (h,\,h')} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

We claim that i is the equalizer of the pair (h, h'). First note that

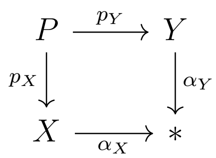
$$(h \circ i, h' \circ i) = (h, h') \circ i = \Delta \circ g = (g, g).$$

So, $h \circ i = g = h' \circ i$. Now assume that $f : Z \to X$ is another morphism in $\mathscr{C}$ satisfying $h \circ f = h' \circ f$. By the universal property of the pullback, there exists a unique morphism $h : Z \to I$ such that the following diagram commutes:

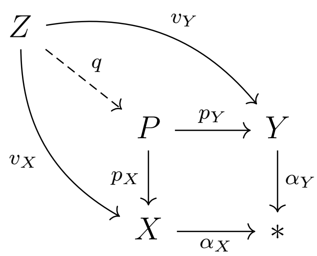


The commutativity of the top triangle in the above diagram tells us that $i = eq(h, h')$. Therefore, $\mathscr{C}$ has equalizers.

To see that $\mathscr{C}$ has products, let $*$ be the terminal object in $\mathscr{C}$. Let X and Y be objects in $\mathscr{C}$. Recall that $\alpha_X : X \to *$ is the terminal map. Form the pullback square



We claim that the triple (P, p_X, p_Y) is a product in $\mathscr{C}$. To this end, assume that Z is an object in $\mathscr{C}$ and $v_X : Z \to X$ and $v_Y : Z \to Y$ are morphisms in $\mathscr{C}$. By the universal property of the pullback, there exists a unique morphism $q : Z \to P$ such that the following diagram commutes:



Therefore, (P, p_X, p_Y) is the product of X and Y and we have constructed products in $\mathscr{C}$. We conclude that $\mathscr{C}$ has equalizers and products.

Finally, assume that $\mathscr{C}$ has equalizers and products. We need to show that $\mathscr{C}$ has all finite limits. Assume that $\mathbf{I}$ is a finite category and $D : \mathbf{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$. Since $\mathscr{C}$ has products then it has all finite products. This will be used to construct a limit of D.

If $u : J \to K$ is a morphism in $\mathbf{I}$ then define $Co(u) = K$ to be the codomain/target of u. Since $\mathscr{C}$ has all finite products then we can form the products

$$\prod_{I \in \mathbf{ob}(\mathbf{I})} D(I) \qquad \text{and} \qquad \prod_{u \in \mathbf{I}} D(Co(u)).$$

These objects in $\mathscr{C}$ have accompanying morphisms

$$\big(\pi_J : \prod_{I \in \mathbf{ob}(\mathbf{I})} D(I) \to D(J)\big)_{J \in \mathbf{ob}(\mathbf{I})}$$

and

$$\big(\rho_v : \prod_{u \in \mathbf{I}} D(Co(u)) \to D(Co(v))\big)_{v \in \mathbf{I}}$$

respectively. Now if $u : J \to K$ is a morphism in $\mathbf{I}$ then define the morphisms

$$s_u = D(u) \circ \pi_J \qquad \text{and} \qquad t_u = \pi_K = \pi_{Co(u)}.$$

The universal property of the product $\prod_{u \in \mathbf{I}} D(Co(u))$ yields two morphisms $s, t : \prod_{I \in \mathbf{ob}(\mathbf{I})} D(I) \to \prod_{u \in \mathbf{I}} D(Co(u))$. Since $\mathscr{C}$ has equalizers then let $p : L \to \prod_{I \in \mathbf{ob}(\mathbf{I})} D(I)$ be the equalizer of s and t. If J is an object in $\mathbf{I}$ then define $p_J = \pi_J \circ p$. We claim that the family of morphisms

$$\big(p_J : L \to D(J)\big)_{I \in \mathbf{ob}(\mathbf{I})}$$

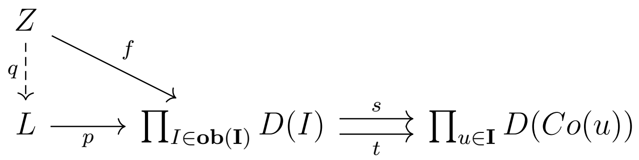
is a limit of D. To see that (p_I) is a cone on D, assume that $u : J \to K$ is a morphism in $\mathbf{I}$. Then

$$\begin{aligned} D(u) \circ p_J &= D(u) \circ \pi_J \circ p \\ &= s_u \circ p = \rho_u \circ s \circ p \\ &= \rho_u \circ t \circ p = t_u \circ p \\ &= \pi_K \circ p = p_K. \end{aligned}$$

Hence, $(p_I)_{I \in \mathbf{I}}$ is a cone on D. To see that it is a limit, assume that

$$\big(f_I : Z \to D(I)\big)_{I \in \mathbf{I}}$$

is another cone on D. By definition of the cone, if u is a morphism in $\mathbf{I}$ then we have $s_u \circ f = t_u \circ f$ and subsequently $s \circ f = t \circ f$. By the universal property of the equalizer, there exists a unique morphism $q : Z \to L$ such that the following diagram commutes:



Therefore if J is an object in $\mathbf{I}$ then

$$p_J \circ q = \pi_J \circ p \circ q = \pi_J \circ f = f_J.$$

So $(p_I)_I$ is a limit on D. Since D was an arbitrary diagram on an arbitrary finite category $\mathbf{I}$ then $\mathscr{C}$ has all finite limits and is finitely complete. This completes the proof. □

We also have a dual characterisation of a finitely cocomplete category, which is proved in a similar fashion to Theorem 4.4.1.

Theorem 4.4.2. *Let $\mathscr{C}$ be a category. The following are equivalent:*

1. *$\mathscr{C}$ is finitely cocomplete,*
2. *$\mathscr{C}$ has pushouts and an initial object,*
3. *$\mathscr{C}$ has coequalizers and coproducts.*

4.5 Cofiltered limits and filtered colimits

Of course, the notion of finitely complete and finitely cocomplete categories can be generalised to include all limits.

Definition 4.5.1. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **complete** if it has all limits. Similarly, we say that $\mathscr{C}$ is **cocomplete** if it has all colimits.

A similar proof to the one in Theorem 4.4.1 yields the following characterisation of complete categories.

Theorem 4.5.1. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is a complete category if and only if $\mathscr{C}$ has equalizers and all arbitrary products.*

To be clear, when we say arbitrary products, we mean that if $\mathbf{I}$ is the small category with no non-zero identity morphisms and whose objects are indexed by a set then the limit of a diagram $D : \mathbf{I} \to \mathscr{C}$ always exists. As usual, we the dual characterisation of cocomplete categories also holds.

Theorem 4.5.2. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is a cocomplete category if and only if $\mathscr{C}$ has coequalizers and all arbitrary coproducts.*

Now, we will give a second characterisation of complete and cocomplete categories, which proceeds through the notions of cofiltered limits and filtered colimits.

Definition 4.5.2. Let $\mathbf{I}$ be a small category. We say that $\mathbf{I}$ is **filtered** if the following three properties are satisfied:

1. $\mathbf{I} \neq \emptyset$.

2. If $i, j \in \mathbf{I}$ are objects then there exists an object $k \in \mathbf{I}$ such that the hom-sets $Hom_{\mathbf{I}}(i, k)$ and $Hom_{\mathbf{I}}(j, k)$ are non-empty.

3. If $f, g : i \to j$ are morphisms in $\mathbf{I}$ then there exists a morphism $h : j \to k$ such that $h \circ f = h \circ g$ (h equalizes f and g).

We say that $\mathbf{I}$ is **cofiltered** if the opposite category $\mathbf{I}^{op}$ is filtered.

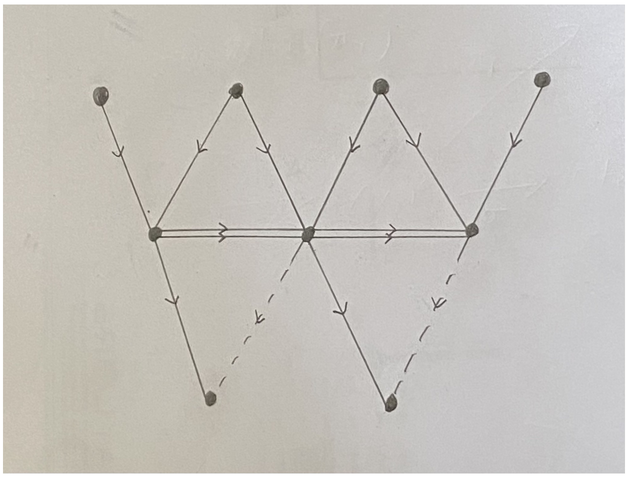


Figure 4.1: A diagram of a filtered small category. Points represent objects and solid lines represent hom-sets. Doubled lines represent a pair of parallel morphisms and dotted lines represent the existence of an equalizing morphism.

Example 4.5.1. As an example of a filtered small category, consider the poset $(\mathbb{Z}_{>0}, \leq)$. This poset can be thought of as a category Θ with $\mathbf{ob}(\Theta) = \mathbb{Z}_{>0}$ and if $x, y \in \mathbb{Z}_{>0}$ then the hom-sets are given by

$$Hom_\Theta(x,y) = \begin{cases} \{*_{x,y}\}, & \text{if } x \leq y, \\ \emptyset, & \text{if } x \not\leq y. \end{cases}$$

First, we have $\mathbb{Z}_{>0} \neq \emptyset$. The third condition is satisfied because if $\alpha, \beta \in \mathbb{Z}_{>0}$ then the cardinality of $Hom_\Theta(\alpha, \beta)$ is at most 1. Therefore, if $f, g : \alpha \to \beta$ then $f = g$ and any morphism from β must equalize f and g.

For the second condition, take $\gamma \in \mathbb{Z}_{>0}$ such that $\gamma \geq \max\{\alpha, \beta\}$. By definition of Θ, the hom-sets $Hom_\Theta(\alpha, \gamma)$ and $Hom_\Theta(\beta, \gamma)$ are non-empty. Therefore, Θ is a filtered small category.

Definition 4.5.3. Let $\mathbf{I}$ be a small category and $\mathscr{C}$ be a category. Let $F : \mathbf{I}^{op} \to \mathscr{C}$ be a functor. A limit of F is called a **cofiltered limit** if $\mathbf{I}$ is filtered. The limit is denoted by

$$\varprojlim_{\mathbf{I}} F.$$

Definition 4.5.4. Let $\mathbf{I}$ be a small category and $\mathscr{C}$ be a category. Let $F : \mathbf{I} \to \mathscr{C}$ be a functor. A colimit of F is called a **filtered colimit** if $\mathbf{I}$ is filtered. The colimit is denoted by

$$\underset{\mathbf{I}}{\underrightarrow{\operatorname{colim}}}\, F.$$

Before we proceed, we will give a few examples of filtered colimits and cofiltered limits.

Example 4.5.2. As an example of a filtered colimit, let $\mathbf{I}$ be a filtered small category and $F : \mathbf{I} \to \mathbf{Set}$ be a functor. We claim that the filtered colimit as a set is

$$\underset{\mathbf{I}}{\underrightarrow{\operatorname{colim}}}\, F = \Big(\coprod_{i \in \mathbf{I}} F(i)\Big)/\sim$$

where $\coprod_{i \in \mathbf{I}} F(i)$ is a disjoint union and $\sim$ is an equivalence relation on $\coprod_{i \in \mathbf{I}} F(i)$ defined in the following manner: If $x \in F(i)$ and $y \in F(j)$ then $x \sim y$ if and only if there exist $\alpha : i \to k$ and $\beta : j \to k$ such that $F(\alpha)(x) = F(\beta)(y)$.

A priori, we have the cocone

$$\big(\pi_I : F(I) \to \underset{\mathbf{I}}{\underrightarrow{\operatorname{colim}}}\, F\big)_{I \in \mathbf{I}}$$

where π_I is the projection map. Now suppose that we have another cocone

$$\big(f_I : F(I) \to S\big)_{I \in \mathbf{I}}$$

of F. We would like to construct a function from $\varinjlim_{\mathbf{I}} F$ to S.

The disjoint union $\coprod_{I \in \mathbf{I}} F(I)$ is a coproduct in **Set**. This means that the following map

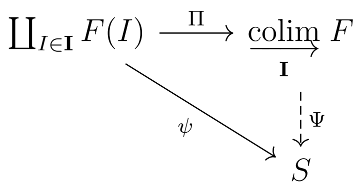
$$\begin{array}{rccc} \psi : & \coprod_{I \in \mathbf{I}} F(I) & \to & S \\ & x \in F(I) & \mapsto & f_I(x) \end{array}$$

is the unique function such that if $I \in \mathbf{I}$ is an object then $f_I = \psi \circ \iota_I$, where $\iota_I : F(I) \to \coprod_{I \in \mathbf{I}} F(I)$ is the inclusion map.

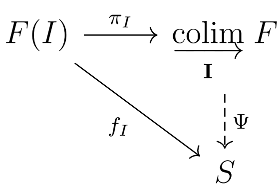
Let $\Pi : \coprod_{I \in \mathbf{I}} F(I) \to \varinjlim_{\mathbf{I}} F$ be the canonical projection map. Assume that $x \in F(I)$ and $y \in F(J)$ are such that $x \sim y$ in $\coprod_{I \in \mathbf{I}} F(I)$. Then, there exist morphisms $\alpha : I \to K$ and $\beta : J \to K$ in $\mathbf{I}$ such that $F(\alpha)(x) = F(\beta)(y)$ in $F(K)$. Since $f_K \circ F(\alpha) = f_I$ and $f_K \circ F(\beta) = f_J$ then

$$f_I(x) = f_K(F(\alpha)(x)) = f_K(F(\beta)(y)) = f_J(y).$$

So, $\psi(x) = \psi(y)$. By the universal property of the quotient, there exists a unique function $\Psi : \varinjlim_{\mathbf{I}} F \to S$ such that the following diagram commutes:



By the constructions of Ψ, Π and ψ, if I is an object in $\mathbf{I}$ then the following diagram must also commute:



This is because $\pi_I = \Pi \circ \iota_I$ and $f_I = \psi \circ \iota_I$. Thus, the collection of morphisms

$$\left(\pi_I : F(I) \to \varinjlim_{\mathbf{I}} F\right)_{I\in\mathbf{I}}$$

is a filtered colimit of F.

Example 4.5.3. As an example of a cofiltered limit, let $\mathbf{I}$ be a filtered small category and $F : \mathbf{I}^{op} \to \mathbf{Set}$ be a functor. We claim that the cofiltered limit of F is the set

$$\varprojlim_{\mathbf{I}} F = \{(x_i)_{i\in\mathbf{I}} \mid F(\alpha)(x_i) = x_j \text{ for some } \alpha : j \to i\} \subseteq \prod_{i\in\mathbf{I}} F(i).$$

Initially, we have the cone

$$\left(\pi_I : \varprojlim_{\mathbf{I}} F \to F(I)\right)_{I\in\mathbf{I}}$$

where $\pi_I((x_i)_{i\in\mathbf{I}}) = x_I$. Now suppose that we have another cone of F

$$\left(h_I : H \to F(I)\right)_{I\in\mathbf{I}}.$$

Define a map ρ from H to $\varprojlim_{\mathbf{I}} F$ by sending $a \in H$ to $(h_I(a))_{I\in\mathbf{I}}$. Note that $(h_I(a))_{I\in\mathbf{I}} \in \varprojlim_{\mathbf{I}} F$ because if $\alpha : j \to i$ is a morphism in $\mathbf{I}$ then $F(\alpha) \circ h_i = h_j$. By definition, it is straightforward to check that if I is an object in $\mathbf{I}$ then $\pi_I \circ \rho = h_I$. Hence, the cone

$$\left(\pi_I : \varprojlim_{\mathbf{I}} F \to F(I)\right)_{I\in\mathbf{I}}$$

is a cofiltered limit in the category $\mathbf{Set}$.

So far, we have concrete characterisations of cofiltered limits and filtered colimits in $\mathbf{Set}$. In the next example, we will give a specific example of a filtered colimit stemming from a particular filtered category.

Example 4.5.4. Let $(\mathbb{Z}_{>0}, \preceq)$ be the poset, where $a \preceq b$ if and only if $a|b$. In a similar manner to Example 4.5.1, $(\mathbb{Z}_{>0}, \preceq)$ is a filtered category. Define the functor $F : (\mathbb{Z}_{>0}, \preceq) \to \mathbf{Ab}$ such that $F(n) = \mathbb{Z}/n\mathbb{Z}$ and

$$\begin{array}{rccc} F(n|m) : & \mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Z}/m\mathbb{Z} \\ & [1]_n & \mapsto & [\frac{m}{n}]_m. \end{array}$$

We will demonstrate that the filtered colimit $\varinjlim F$ is isomorphic to the abelian group $\mathbb{Q}/\mathbb{Z}$.

If $n \in \mathbb{Z}_{>0}$ then define the abelian group morphisms

$$\begin{array}{rccc} \alpha_n : & F(n) = \mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Q}/\mathbb{Z} \\ & [1]_n & \mapsto & [\frac{1}{n}]. \end{array}$$

To show: (a) $\{\alpha_n\}_{n\in\mathbb{Z}_{>0}}$ is a cocone of F.

(a) Assume that $m, n \in \mathbb{Z}_{>0}$ such that $n|m$. We compute directly that

$$(\alpha_m \circ F(n|m))([1]_n) = \alpha_m([\frac{m}{n}]_m) = [\frac{1}{n}] = \alpha_n([1]_n).$$

So $\alpha_m \circ F(n|m) = \alpha_n$ and consequently, $\{\alpha_n\}_{n\in\mathbb{Z}_{>0}}$ is a cocone of F.

Now let $\{p_n : F(n) \to \varinjlim F\}_{n\in\mathbb{Z}_{>0}}$ be a colimit of F. Then there exists a unique morphism $f : \varinjlim F \to \mathbb{Q}/\mathbb{Z}$ such that if $n \in \mathbb{Z}_{>0}$ then $f \circ p_n = \alpha_n$.

To show: (b) f is an isomorphism of abelian groups.

(b) Define the map g by

$$\begin{array}{rccc} g : & \mathbb{Q}/\mathbb{Z} & \to & \varinjlim F \\ & [\frac{1}{n}] & \mapsto & p_n([1]_n) \end{array}$$

Note that if $m \in \mathbb{Z}_{>0}$ then

$$g([\frac{m}{n}]) = mg([\frac{1}{n}]) = mp_n([1]_n) = p_n([m]_n).$$

To see that $m \in \mathbb{Z}_{>0}$, suppose that $[\frac{m_1}{n_1}] = [\frac{m_2}{n_2}]$ in $\mathbb{Q}/\mathbb{Z}$. Then, $n_1 = n_2$ and $m_2 = m_1 + kn_1$ for some $k \in \mathbb{Z}$. We compute directly that

$$g([\frac{m_2}{n_2}]) = p_{n_1}([m_1 + kn_1]_{n_1}) = p_{n_1}([m_1]_{n_1}) = g([\frac{m_1}{n_1}]).$$

Therefore g is well-defined.

To see that g is an abelian group morphism, assume that $[\frac{a_1}{b_1}], [\frac{a_2}{b_2}] \in \mathbb{Q}/\mathbb{Z}$. Then,

$$\begin{aligned} g([\frac{a_1}{b_1}]+[\frac{a_2}{b_2}]) &= g([\frac{a_1b_2+a_2b_1}{b_1b_2}]) \\ &= p_{b_1b_2}([a_1b_2+a_2b_1]_{b_1b_2}) \\ &= p_{b_1b_2}([a_1b_2]_{b_1b_2}) + p_{b_1b_2}([a_2b_1]_{b_1b_2}) \\ &= g([\frac{a_1b_2}{b_1b_2}]) + g([\frac{a_2b_1}{b_1b_2}]) \\ &= g([\frac{a_1}{b_1}]) + g([\frac{a_2}{b_2}]). \end{aligned}$$

Hence, g is an abelian group morphism.

To see that g is the inverse of f, we compute directly that

$$((g \circ f) \circ p_n)([1]_n) = (g \circ \alpha_n)([1]_n) = g([\frac{1}{n}]) = p_n([1]_n).$$

So, $(g \circ f) \circ p_n = id_{\varinjlim F} \circ p_n$. By the universal property of the colimit, we find that $g \circ f = id_{\varinjlim F}$. We also have

$$(f \circ g)([\frac{m}{n}]) = (f \circ p_n)([m]_n) = \alpha_n([m]_n) = [\frac{m}{n}].$$

Hence, $f \circ g = id_{\mathbb{Q}/\mathbb{Z}}$. Therefore, f is an isomorphism of abelian groups.

From part (b), we conclude that $\varinjlim F$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Tying all the concepts introduced in this section is the following characterisation of complete categories.

Theorem 4.5.3. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is complete if and only if $\mathscr{C}$ is finitely complete and has cofiltered limits.*

Proof. Assume that $\mathscr{C}$ is a category. If $\mathscr{C}$ is complete then it has all limits, including finite limits and cofiltered limits. Conversely, assume that $\mathscr{C}$ is finitely complete and has cofiltered limits. By Theorem 4.5.1, it suffices to show that $\mathscr{C}$ has all arbitrary products.

To do this, let $\mathbf{I}$ be a set (which can be thought of as a category with no non-identity morphisms) and $F : \mathbf{I} \to \mathscr{C}$ be a functor. Let $\mathbf{I}^+$ be the category whose objects are finite subsets of $\mathbf{I}$ and whose morphisms are inclusions of finite subsets. By definition, $\mathbf{I}^+$ is a small category.

To show: (a) $\mathbf{I}^+$ is filtered.

(a) First note that $\mathbf{I}^+$ is non-empty because it contains the set $\mathbf{I}$. Next assume that J and K are finite subsets of $\mathbf{I}$ (objects in $\mathbf{I}^+$). Then $L = J \cup K$ is a finite subset of $\mathbf{I}$ such that

$$Hom_{\mathbf{I}^+}(J, L) \neq \emptyset \qquad \text{and} \qquad Hom_{\mathbf{I}^+}(K, L) \neq \emptyset.$$

Finally, if we have two morphisms $f, g : S_1 \to S_2$ in $\mathbf{I}^+$ then $f = g$ because they are both equal to the inclusion map $S_1 \hookrightarrow S_2$. Hence any morphism $h : S_2 \to T$ in $\mathbf{I}^+$ must satisfy $h \circ f = h \circ g$. So $\mathbf{I}^+$ is filtered.

Since $\mathbf{I}^+$ is a filtered category then the opposite category $(\mathbf{I}^+)^{op}$ is cofiltered. Define the functor F^+ by

$$\begin{array}{rccc} F^+ : & (\mathbf{I}^+)^{op} & \to & \mathscr{C} \\ & J & \mapsto & \varprojlim_J F|_J \end{array}$$

This is well-defined on objects because $\mathscr{C}$ has finite limits. Now let $u : J \to K$ be a morphism in $(\mathbf{I}^+)^{op}$. By definition of $\mathbf{I}^+$, $K \subseteq J$ and we have associated limits

$$\left(\sigma_{J,i} : \varprojlim_J F|_J \to F(i)\right)_{i \in J}$$

and

$$\left(\sigma_{K,i} : \varprojlim_K F|_K \to F(i)\right)_{i \in K}.$$

Since $K \subseteq J$ then $(\sigma_{J,i})_{i \in K}$ is a cone on the restricted functor $F|_K$. The morphism $F^+(u)$ is created by the universal property of the limit $(\sigma_{K,i})_{i \in K}$ — $F^+(u)$ is the unique morphism in $\mathscr{C}$ such that if $i \in K$ then

$$\sigma_{K,i} \circ F^+(u) = \sigma_{J,i}.$$

It is straightforward but tedious to check that F^+ is a functor.

To show: (b) The cofiltered limit $\varprojlim_{(\mathbf{I}^+)^{op}} F^+$ is the product associated to the set $\mathbf{I}$.

(b) Firstly, observe that if K is a finite subset of $\mathbf{I}$ then we have limits

$$\left(p_J : \varprojlim_{(\mathbf{I}^+)^{op}} F^+ \to \varprojlim_J F|_J\right)_{J \in (\mathbf{I}^+)^{op}}$$

and

$$\left(\sigma_{K,i} : \varprojlim_{K} F|_K \to F(i)\right)_{i\in K}$$

of F^+ and $F|_K$ respectively. These limits exist because $\mathscr{C}$ has cofiltered limits and finite limits respectively. We want to show that $\varprojlim_{(\mathbf{I}^+)^{op}} F^+$ is also the limit of F. If $i \in \mathbf{I}$ then we have the family of morphisms.

$$\left(\sigma_{\{i\},i} \circ p_{\{i\}} : \varprojlim_{(\mathbf{I}^+)^{op}} F^+ \to F(i)\right)_{i\in \mathbf{I}}. \tag{4.3}$$

To see that this is a cone of F, assume that $v : \{i\} \to \{j\}$ is a morphism in $\mathbf{I}$. Since $\mathbf{I}$ is a set then it has no non-identity morphisms. So $i = j$, $v = id_{\{i\}}$ and $\sigma_{\{i\},i} = F(v) \circ \sigma_{\{i\},i}$.

To see that the cone in equation (4.3) is a limit of F, assume that

$$\left(f_i : V \to F(i)\right)_{i\in \mathbf{I}}$$

is another cone of F. If J is a finite subset of $\mathbf{I}$ then by the universal property of the limit $(\sigma_{J,i})_{i\in J}$, there exists a unique morphism $\alpha_J : V \to \varprojlim_{J} F|_J$ such that if $i \in J$ then $\sigma_{J,i} \circ \alpha_J = f_i$.

Next, form the following family of morphisms:

$$\left(\alpha_J : V \to \varprojlim_{J} F|_J\right)_{J\in(\mathbf{I}^+)^{op}}.$$

To see that this is a cone of F^+, assume that $u : J \to K$ is a morphism in $(\mathbf{I}^+)^{op}$ so that $K \subseteq J$. By definition of the morphism $F^+(u) : F^+(J) \to F^+(K)$, we compute directly that if $i \in K$ then

$$\sigma_{K,i} \circ (F^+(u) \circ \alpha_J) = \sigma_{J,i} \circ \alpha_J = f_i.$$

By uniqueness, we find that $F^+(u) \circ \alpha_J = \alpha_K$. Therefore the family of morphisms $(\alpha_J)_J$ is a cone of F^+.

By the universal property of the limit once again, there exists a unique morphism $\beta : V \to \varprojlim_{(\mathbf{I}^+)^{op}} F^+$ such that if J is a finite subset of $\mathbf{I}$ then $p_J \circ \beta = \alpha_J$. Now if $i \in \mathbf{I}$ then $\{i\}$ is a finite subset of $\mathbf{I}$ and

$$\sigma_{\{i\},i} \circ p_{\{i\}} \circ \beta = \sigma_{\{i\},i} \circ \alpha_i = f_i.$$

We conclude that the cone of F in equation (4.3) is the desired limit of F. Since $\mathbf{I}$ was an arbitrary set then we have constructed arbitrary products in $\mathscr{C}$ and by Theorem 4.5.1, $\mathscr{C}$ is a complete category. □

Dually, we have the following characterisation of cocomplete categories.

Theorem 4.5.4. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is cocomplete if and only if $\mathscr{C}$ is finitely cocomplete and has filtered colimits.*

4.6 Interpreting limits as functors

It is not difficult to show from the definition of a limit that it is unique up to a unique isomorphism, provided that it exists.

Theorem 4.6.1. *Let $\boldsymbol{I}$ be a small category and $\mathscr{C}$ be a category. Let $D : \boldsymbol{I} \to \mathscr{C}$ be a diagram and*

$$\big(p_I : L_1 \to D(I)\big)_{I \in \boldsymbol{I}} \qquad \textit{and} \qquad \big(q_I : L_2 \to D(I)\big)_{I \in \boldsymbol{I}}$$

be two limits of D. Then there exists a unique isomorphism $\psi : L_1 \to L_2$.

Proof. Assume that $\mathbf{I}$ is a small category, $\mathscr{C}$ is a category and $D : \mathbf{I} \to \mathscr{C}$ is a diagram. Assume that $(p_I)_{I\in\mathbf{I}}$ and $(q_I)_{I\in\mathbf{I}}$ are both limits of D, defined as in the statement of the theorem. By the universal property of the limit, there exist unique morphisms $\psi : L_1 \to L_2$ and $\varphi : L_2 \to L_1$ such that if $I \in \mathbf{I}$ then

$$q_I \circ \psi = p_I \qquad \text{and} \qquad p_I \circ \varphi = q_I.$$

Again by the universal property of the limit, the identity morphisms id_{L_1} and id_{L_2} are the unique morphisms such that if $I \in \mathbf{I}$ then $p_I \circ id_{L_1} = p_I$ and $q_I \circ id_{L_2} = q_I$. But $p_I \circ (\varphi \circ \psi) = p_I$ and $q_I \circ (\psi \circ \varphi) = q_I$. By uniqueness, $\varphi \circ \psi = id_{L_1}$ and $\psi \circ \varphi = id_{L_2}$. This demonstrates that ψ is a unique isomorphism from L_1 to L_2, completing the proof. □

By a very similar proof, a colimit, if it exists, is unique up to a unique isomorphism.

Theorem 4.6.2. *Let $\boldsymbol{I}$ be a small category and $\mathscr{C}$ be a category. Let $D : \boldsymbol{I} \to \mathscr{C}$ be a diagram and*

$$\big(i_I : D(I) \to C_1\big)_{I \in \boldsymbol{I}} \qquad \textit{and} \qquad \big(j_I : D(I) \to C_2\big)_{I \in \boldsymbol{I}}$$

be two colimits of D. Then there exists a unique isomorphism $\psi : C_1 \to C_2$.

The universal property of the limit can also be used to construct unique morphisms between limits of different diagrams from the same small category.

Lemma 4.6.3. *Let $\mathscr{C}$ be a category and $\boldsymbol{I}$ be a small category. Let $D : \boldsymbol{I} \to \mathscr{C}$ be a diagram and*

$$\big(p_I : L \to D(I)\big)_{I \in \boldsymbol{I}}$$

be a limit on D. If $h, h' : A \to L$ are morphisms satisfying $p_I \circ h = p_I \circ h'$ for $I \in \boldsymbol{I}$ then $h = h'$.

Proof. Assume that $\mathscr{C}$ is a category and $\mathbf{I}$ is a small category. Assume that $D : \mathbf{I} \to \mathscr{C}$ is a diagram and $(p_I)_{I \in \mathbf{I}}$ is a limit on D. Assume that h, h' are morphisms from A to L such that if $I \in \mathbf{I}$ then $p_I \circ h = p_I \circ h'$. By the universal property of the limit L, there exists a unique morphism $f : A \to L$ such that if $I \in \mathbf{I}$ then $p_I \circ f = p_I \circ h = p_I \circ h'$. By uniqueness of f, $h = h'$. □

Theorem 4.6.4. *Let $\boldsymbol{I}$ be a small category and $\mathscr{C}$ be a category. Let $D, D' : \boldsymbol{I} \to \mathscr{C}$ be diagrams and $\alpha : D \Rightarrow D'$ be a natural transformation. Let*

$$\big(p_I : \varprojlim_{\boldsymbol{I}} D \to D(I)\big)_{I \in \boldsymbol{I}} \qquad \text{and} \qquad \big(p'_I : \varprojlim_{\boldsymbol{I}} D' \to D'(I)\big)_{I \in \boldsymbol{I}}$$

be the limits of D and D' respectively. Then, there exists a unique morphism $\varprojlim_{\boldsymbol{I}} \alpha : \varprojlim_{\boldsymbol{I}} D \to \varprojlim_{\boldsymbol{I}} D'$ such that if $I \in \boldsymbol{I}$ then the following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} \varprojlim_{\boldsymbol{I}} D & \xrightarrow{p_I} & D(I) \\ {\scriptstyle \varprojlim_{\boldsymbol{I}} \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_I} \\ \varprojlim_{\boldsymbol{I}} D' & \xrightarrow[p'_I]{} & D'(I) \end{array}$$

Moreover, if we have two cones

$$\big(f_I : A \to D(I)\big)_{I \in \boldsymbol{I}} \qquad \text{and} \qquad \big(f'_I : A' \to D'(I)\big)_{I \in \boldsymbol{I}}$$

and a morphism $s : A \to A'$ which makes the following diagram commute for $I \in \boldsymbol{I}$

$$\begin{array}{ccc} A & \xrightarrow{f_I} & D(I) \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_I} \\ A' & \xrightarrow[f'_I]{} & D'(I) \end{array}$$

then the square below also commutes:

$$\begin{array}{ccc} A & \xrightarrow{\overline{f}} & \varprojlim_{\mathbf{I}} D \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \varprojlim_{\mathbf{I}} \alpha} \\ A' & \xrightarrow[\overline{f'}]{} & \varprojlim_{\mathbf{I}} D' \end{array}$$

Proof. Assume that $\mathbf{I}$ is a small category and $\mathscr{C}$ is a category. Assume that $D, D' : \mathbf{I} \to \mathscr{C}$ be diagrams and $\alpha : D \Rightarrow D'$ is a natural transformation.

We will construct a unique morphism from $\varprojlim_{\mathbf{I}} D$ to $\varprojlim_{\mathbf{I}} D'$. Making use of the natural transformation α_I, observe that we have the following cone on D':

$$\big(\alpha_I \circ p_I : \varprojlim_{\mathbf{I}} D \to D'(I)\big)_{I \in \mathbf{I}}$$

By the universal property of the limit $\varprojlim_{\mathbf{I}} D'$, there exists a unique morphism $\varprojlim_{\mathbf{I}} \alpha : \varprojlim_{\mathbf{I}} D \to \varprojlim_{\mathbf{I}} D'$ such that if $I \in \mathbf{I}$ then the following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} \varprojlim_{\mathbf{I}} D & \xrightarrow{p_I} & D(I) \\ {\scriptstyle \varprojlim_{\mathbf{I}} \alpha}\downarrow & & \downarrow{\scriptstyle \alpha_I} \\ \varprojlim_{\mathbf{I}} D' & \xrightarrow[p'_I]{} & D'(I) \end{array}$$

Next, assume that we have two cones

$$\big(f_I : A \to D(I)\big)_{I \in \mathbf{I}} \qquad \text{and} \qquad \big(f'_I : A' \to D'(I)\big)_{I \in \mathbf{I}}$$

and a morphism $s : A \to A'$ satisfying $f'_1 \circ s = \alpha_I \circ f_I$. By the universal property of the limit, we obtain unique morphisms $\overline{f} : A \to \varprojlim_{\mathbf{I}} D$ and $\overline{f'} : A' \to \varprojlim_{\mathbf{I}} D'$. Now observe that if $I \in \mathbf{I}$ then

$$\begin{aligned} p'_I \circ (\varprojlim_{\mathbf{I}} \alpha) \circ \overline{f} &= (p'_I \circ \varprojlim_{\mathbf{I}} \alpha) \circ \overline{f} \\ &= \alpha_I \circ p_I \circ \overline{f} \\ &= \alpha_I \circ f_I = f'_I \circ s \\ &= p'_I \circ \overline{f'} \circ s. \end{aligned}$$

By Lemma 4.6.3, we find that $\overline{f'} \circ s = \varprojlim_{\mathbf{I}} \alpha \circ \overline{f}$ as required. □

With Theorem 4.6.4, we can now define limits of diagrams from a small category as a functor.

Definition 4.6.1. Let $\mathbf{I}$ be a small category and $\mathscr{C}$ be a category which has all limits of shape $\mathbf{I}$. The **limit functor** $\lim_{\mathbf{I}}$ is defined as follows:

$$\begin{array}{rccc} \lim_{\mathbf{I}}: & \mathcal{F}(\mathbf{I},\mathscr{C}) & \to & \mathscr{C} \\ & D & \mapsto & \varprojlim_{\mathbf{I}} D \\ & \phi : D \Rightarrow D' & \mapsto & \varprojlim_{\mathbf{I}} \phi. \end{array}$$

Here, the morphism $\varprojlim_{\mathbf{I}} \phi$ is the unique morphism constructed in Theorem 4.6.4.

The fact that limit functors are actually functors follows from Theorem 4.6.4. We will now show that a limit functor is a right adjoint functor. The reason we might expect this is due to the original definition of a limit as a collection of morphisms.

Definition 4.6.2. Let $\mathbf{I}$ be a small category and $\mathscr{C}$ be a category with all limits of shape $\mathbf{I}$. Define the **diagonal functor** $\Delta : \mathscr{C} \to \mathcal{F}(\mathbf{I},\mathscr{C})$ as follows: If A is an object in $\mathscr{C}$ then $\Delta(A)$ is the constant functor

$$\begin{array}{rccc} \Delta(A): & \mathbf{I} & \to & \mathscr{C} \\ & I & \to & A \\ & \alpha : I \to J & \mapsto & id_A. \end{array}$$

If $f : A \to B$ is a morphism in $\mathscr{C}$ then $\Delta(f) : \Delta(A) \Rightarrow \Delta(B)$ is the natural transformation given by the collection of morphisms

$$\big(\Delta(f)_I : A \to B\big)_{I\in\mathbf{I}} = \big(f : A \to B\big)_{I\in\mathbf{I}}.$$

Theorem 4.6.5. *Let $\boldsymbol{I}$ be a small category and $\mathscr{C}$ be a category with all limits of shape $\boldsymbol{I}$. Let $\lim_{\boldsymbol{I}} : \mathcal{F}(\boldsymbol{I},\mathscr{C}) \to \mathscr{C}$ and $\Delta : \mathscr{C} \to \mathcal{F}(\boldsymbol{I},\mathscr{C})$ be the limit functor and the diagonal functor respectively. Then the pair $(\Delta, \lim_{\boldsymbol{I}})$ is an adjoint pair of functors.*

Proof. Assume that $\mathbf{I}$ is a small category and $\mathscr{C}$ is a category with all limits of shape $\mathbf{I}$. Let A be an object in $\mathscr{C}$ and $F : \mathbf{I} \to \mathscr{C}$ be an object in the functor category $\mathcal{F}(\mathbf{I},\mathscr{C})$. Assume that

$$\alpha \in Hom_{\mathcal{F}(\mathbf{I},\mathscr{C})}(\Delta(A), F).$$

Then α is a natural transformation from $\Delta(A)$ to F. This is a collection of morphisms

$$\big(\alpha_I : \Delta(A)(I) = A \to F(I)\big)_{I\in\mathbf{I}}.$$

such that if $f : I \to J$ is a morphism in $\mathbf{I}$ then

$$\alpha_J = \alpha_J \circ id_A = \alpha_J \circ \Delta(A)(f) = F(f) \circ \alpha_I.$$

Therefore $(\alpha_I)_{I\in\mathbf{I}}$ is a cone on F. Now let

$$\big(p_I : \varprojlim_{\mathbf{I}} F \to F(I)\big)_{I\in\mathbf{I}}.$$

be the limit on F. By the universal property of the limit, there exists a unique morphism $\tilde{\alpha} : A \to \varprojlim_{\mathbf{I}} F$ such that if I is an object in $\mathbf{I}$ then $p_I \circ \tilde{\alpha} = \alpha_I$. Now define the map

$$\begin{array}{rccc} \tau_{A,F} : & Hom_{\mathcal{F}(\mathbf{I},\mathscr{C})}(\Delta(A), F) & \to & Hom_{\mathscr{C}}(A, \lim_{\mathbf{I}} F) \\ & \alpha & \mapsto & \tilde{\alpha} \end{array}$$

We will now show that $\tau_{A,F}$ is a bijection. To see that $\tau_{A,F}$ is injective, assume that $\alpha, \beta : \Delta(A) \Rightarrow F$ are natural transformations satisfying $\tau_{A,F}(\alpha) = \tau_{A,F}(\beta)$. By construction of $\tilde{\alpha}$ and $\tilde{\beta}$, if I is an object in $\mathbf{I}$ then

$$\alpha_I = p_I \circ \tau_{A,F}(\alpha) = p_I \circ \tau_{A,F}(\beta) = \beta_I.$$

Hence $\alpha = \beta$ and $\tau_{A,F}$ is injective. Next assume that we have a morphism $g : A \to \lim_{\mathbf{I}} F$. We have a natural transformation $\gamma : \Delta(A) \Rightarrow F$ given by the collection of morphisms

$$\big(\gamma_I = p_I \circ g : A \to F(I)\big)_{I\in\mathbf{I}}.$$

Note that $\tau_{A,F}(\gamma)$ is the unique morphism such that if I is an object in $\mathbf{I}$ then $p_I \circ \tau_{A,F}(\gamma) = \gamma_I = p_I \circ g$. By uniqueness, $\tau_{A,F}(\gamma) = g$ and $\tau_{A,F}$ is surjective. Consequently, $\tau_{A,F}$ is bijective and $(\Delta, \lim_{\mathbf{I}})$ is an adjoint pair of functors. □

Dually, we can repeat what was done in this section so far for colimits and obtain the *colimit functor*. Analogously to Theorem 4.6.5, the colimit functor $\operatorname{colim}_{\mathbf{I}} : \mathcal{F}(\mathbf{I}, \mathscr{C}) \to \mathscr{C}$ is left adjoint to the diagonal functor $\Delta : \mathscr{C} \to \mathcal{F}(\mathbf{I}, \mathscr{C})$.

We will end this section with one particular application of Theorem 4.6.5; we will show that limits commute and thus can be interchanged. In the proof we will use the fact that the category of small categories **Cat** is finitely complete without proof. By Theorem 4.4.1, one can prove this by constructing products and equalizers in **Cat**.

Theorem 4.6.6. *Let $\boldsymbol{I}$, $\boldsymbol{J}$ be small categories and $\mathscr{C}$ be a category which has limits of shapes $\boldsymbol{I}$ and $\boldsymbol{J}$. Then $\mathscr{C}$ has limits of shape $\boldsymbol{I}\times\boldsymbol{J}$ and if $F:\boldsymbol{I}\times\boldsymbol{J}\to\mathscr{C}$ is a diagram then*

$$\varprojlim_{\boldsymbol{I}\times\boldsymbol{J}} F \cong \varprojlim_{\boldsymbol{I}}\varprojlim_{\boldsymbol{J}} F \cong \varprojlim_{\boldsymbol{J}}\varprojlim_{\boldsymbol{I}} F.$$

Proof. Assume that $\mathbf{I}$, $\mathbf{J}$ and $\mathscr{C}$ are the categories defined in the statement of theorem. Let $F:\mathbf{I}\times\mathbf{J}\to\mathscr{C}$ be a diagram. We will think of F as a functor from $\mathbf{I}$ to the functor category $\mathcal{F}(\mathbf{J},\mathscr{C})$.

Define the diagonal functors

$$\Delta_1:\mathscr{C}\to\mathcal{F}(\mathbf{I},\mathscr{C}),$$

$$\Delta_2:\mathcal{F}(\mathbf{I},\mathscr{C})\to\mathcal{F}(\mathbf{J},\mathcal{F}(\mathbf{I},\mathscr{C}))$$

and

$$\Delta_3:\mathscr{C}\to\mathcal{F}(\mathbf{I}\times\mathbf{J},\mathscr{C}).$$

If A is an object in $\mathscr{C}$ then by Theorem 4.6.5, we have the sequence of natural isomorphisms

$$\begin{aligned} Hom_{\mathscr{C}}(A,\varprojlim_{\mathbf{I}}\varprojlim_{\mathbf{J}} F) &\cong Hom_{\mathcal{F}(\mathbf{I},\mathscr{C})}(\Delta_1(A),\varprojlim_{\mathbf{J}} F) \\ &\cong Hom_{\mathcal{F}(\mathbf{J},\mathcal{F}(\mathbf{I},\mathscr{C}))}((\Delta_2\circ\Delta_1)(A),F) \\ &\cong Hom_{\mathcal{F}(\mathbf{I}\times\mathbf{J},\mathscr{C})}(\Delta_3(A),F). \end{aligned}$$

By Theorem 4.6.5, the limit $\varprojlim_{\mathbf{I}\times\mathbf{J}} F$ exists and

$$\varprojlim_{\mathbf{I}}\varprojlim_{\mathbf{J}} F \cong \varprojlim_{\mathbf{I}\times\mathbf{J}} F.$$

However by interchanging the roles of $\mathbf{I}$ and $\mathbf{J}$ in the above argument, we deduce that

$$\varprojlim_{\mathbf{J}\times\mathbf{I}} F \cong \varprojlim_{\mathbf{I}}\varprojlim_{\mathbf{J}} F \cong \varprojlim_{\mathbf{I}\times\mathbf{J}} F.$$

as required. □

Once again, a dual result to Theorem 4.6.6 also holds.

Theorem 4.6.7. *Let $\boldsymbol{I}$, $\boldsymbol{J}$ be small categories and $\mathscr{C}$ be a category which has colimits of shapes $\boldsymbol{I}$ and $\boldsymbol{J}$. Then $\mathscr{C}$ has colimits of shape $\boldsymbol{I} \times \boldsymbol{J}$ and if $F : \boldsymbol{I} \times \boldsymbol{J} \to \mathscr{C}$ is a diagram then*

$$colim_{\boldsymbol{I}\times\boldsymbol{J}} F \cong colim_{\boldsymbol{I}}\ colim_{\boldsymbol{J}} F \cong colim_{\boldsymbol{J}}\ colim_{\boldsymbol{I}} F.$$

In general, limits do not commute with colimits.

Chapter 5

Adjoint pairs and (co)limits

5.1 Preservation and reflection

This chapter is dedicated to examining the interactions between adjoint pairs of functors and limits/colimits. Our first result tells us that left adjoint functors preserve colimits whereas right adjoint functors preserve limits.

Theorem 5.1.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that (F, G) is an adjoint pair. Then, F preserves colimits and G preserves limits.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories. Assume that (F, G) is a pair of adjoint functors. If A is an object in $\mathscr{C}$ and B is an object in $\mathscr{D}$ then we have the bijection

$$\tau_{A,B} : Hom_{\mathscr{D}}(F(A), B) \to Hom_{\mathscr{C}}(A, G(B))$$

which is natural in both A and B. Recall that this means that the following diagram commutes:

$$\begin{array}{ccccc}
Hom_{\mathscr{D}}(F(A'), B) & \xrightarrow{F(f)^*} & Hom_{\mathscr{D}}(F(A), B) & \xrightarrow{g_*} & Hom_{\mathscr{D}}(F(A), B') \\
\downarrow{\scriptstyle \tau_{A',B}} & & \downarrow{\scriptstyle \tau_{A,B}} & & \downarrow{\scriptstyle \tau_{A,B'}} \\
Hom_{\mathscr{C}}(A', G(B)) & \xrightarrow[f^*]{} & Hom_{\mathscr{C}}(A, G(B)) & \xrightarrow[G(g)_*]{} & Hom_{\mathscr{C}}(A, G(B'))
\end{array}$$

By duality, it suffices to show that F preserves colimits. Suppose that $\mathbf{I}$ is a small category and $D : \mathbf{I} \to \mathscr{C}$ is a functor. Suppose that

$$\big(i_I : D(I) \to C\big)_{I \in \mathbf{I}}$$

is a colimit of D. We claim that

$$\big(F(i_I) : F(D(I)) \to F(C)\big)_{I \in \mathbf{I}}$$

is a colimit of $F \circ D$. Note that $(F(i_I))_{I\in\mathbf{I}}$ is a cocone of $F \circ D$ because $(i_I)_{I\in\mathbf{I}}$ is a cocone of D. Now suppose that

$$\big(k_I : F(D(I)) \to T\big)_{I \in \mathbf{I}}$$

is another cocone of $F \circ D$. If $u : I \to J$ is a morphism in $\mathbf{I}$ then $k_J \circ F(D(u)) = k_I$ and $F(D(u))^*(k_J) = k_I$. If we apply the bijection $\tau_{D(I),T}$, we find that

$$\begin{aligned}
\tau_{D(I),T}(k_I) &= (\tau_{D(I),T} \circ F(D(u))^*)(k_J) = (D(u)^* \circ \tau_{D(J),T})(k_J) \\
&= \tau_{D(J),T}(k_J) \circ D(u).
\end{aligned}$$

This tells us that the collection of morphisms

$$\big(\tau_{D(I),T}(k_I) : D(I) \to G(T)\big)_{I \in \mathbf{I}}$$

is a cocone of D. By the universal property of the colimit, there exists a unique morphism $\tilde{k} : C \to G(T)$ such that if $I \in \mathbf{I}$ then $\tilde{k} \circ i_I = \tau_{D(I),T}(k_I)$. So, $(i_I)^*(\tilde{k}) = \tau_{D(I),T}(k_I)$. By taking the inverse $\tau^{-1}_{D(I),T}$, we obtain

$$\begin{aligned}
k_I &= \tau^{-1}_{D(I),T}(\tau_{D(I),T}(k_I)) = \tau^{-1}_{D(I),T}((i_I)^*(\tilde{k})) \\
&= (\tau^{-1}_{D(I),T} \circ (i_I)^*)(\tilde{k}) = (F(i_I)^* \circ \tau^{-1}_{C,T})(\tilde{k}) \\
&= \tau^{-1}_{C,T}(\tilde{k}) \circ F(i_I).
\end{aligned}$$

So $\tau^{-1}_{C,T}(\tilde{k}) : F(C) \to T$ is the desired unique morphism and consequently

$$\big(F(i_I) : F(D(I)) \to F(C)\big)_{I \in \mathbf{I}}$$

is a colimit of $F \circ D$. We conclude that F preserves colimits. □

Definition 5.1.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Let $\mathbf{I}$ be a small category. We say that F **preserves limits** if the following statement is satisfied: If $D : \mathbf{I} \to \mathscr{C}$ is a diagram and

$$\big(p_I : L \to D(I)\big)_{I \in \mathbf{I}}$$

defines a limit of D then $(F(p_I))_{I\in\mathbf{I}}$ is a limit of $F\circ D$.

We say that F **reflects limits** if the following statement is satisfied: If $D:\mathbf{I}\to\mathscr{C}$ is a diagram,

$$\big(q_I : L\to D(I)\big)_{I\in\mathbf{I}}$$

is a cone of D and $(F\circ q_I)_{I\in\mathbf{I}}$ is a limit of $F\circ D$ then $(q_I)_{I\in\mathbf{I}}$ is a limit of D.

The dual definitions apply to functors preserving and reflecting colimits.

Example 5.1.1. Let $\mathscr{C}$ be a locally small category. Recall the Yoneda embedding from Definition 3.1.2. Assume that $\mathbf{I}$ is a small category and $J:\mathbf{I}\to\mathscr{C}$ is a diagram. Let $\Delta:\mathscr{C}\to\mathcal{F}(\mathbf{I},\mathscr{C})$ and $\Delta':\mathbf{Set}\to\mathcal{F}(\mathbf{I},\mathbf{Set})$ be diagonal functors.

Let $*=\{*\}$ denote the singleton set. Define the contravariant functor

$$\begin{array}{rccc} Hom_{\mathscr{C}}(J,X): & \mathbf{I} & \to & \mathbf{Set} \\ & I & \mapsto & Hom_{\mathscr{C}}(J(I),X) \\ & j:I\to K & \mapsto & J(j)_* \end{array}$$

where $J(j)_*$ is a map from $Hom_{\mathscr{C}}(J(K),X)$ to $Hom_{\mathscr{C}}(J(I),X)$ defined by precomposition by $J(j)$. Then, a morphism $\beta:\Delta'(*)\Rightarrow Hom_{\mathscr{C}}(J,X)$ in the functor category $\mathcal{F}(\mathbf{I}^{op},\mathscr{C})$ is the family of morphisms

$$\big(\beta'_I : *=\Delta'(*)(I)\to Hom_{\mathscr{C}}(J(I),X)=Hom_{\mathscr{C}}(J,X)(I)\big)_{I\in\mathbf{I}}.$$

This is equivalent to having the family of morphisms

$$\big(\beta_I : J(I)\to\Delta(X)(I)=X\big)_{I\in\mathbf{I}}$$

in $\mathscr{C}$ where $\beta_I=\beta'_I(*)$. One can verify that this gives rise to the natural isomorphism

$$Hom_{\mathcal{F}(\mathbf{I},\mathscr{C})}(J,\Delta(X))\cong Hom_{\mathcal{F}(\mathbf{I}^{op},\mathbf{Set})}(\Delta'(*),Hom_{\mathscr{C}}(J,X)).$$

Now observe that we have the following chain of isomorphisms which are natural in X:

$$\begin{aligned} Hom_{\mathbf{Set}}(Hom_{\mathscr{C}}(\operatorname{colim}_{\mathbf{I}} J,X),*) &\cong Hom_{\mathscr{C}}(\operatorname{colim}_{\mathbf{I}} J,X) \\ &\cong Hom_{\mathcal{F}(\mathbf{I},\mathscr{C})}(J,\Delta(X)) \\ &\cong Hom_{\mathcal{F}(\mathbf{I}^{op},\mathbf{Set})}(\Delta'(*),Hom_{\mathscr{C}}(J,X)) \\ &\cong Hom_{\mathbf{Set}}(\varprojlim_{\mathbf{I}} Hom_{\mathscr{C}}(J,X),*). \end{aligned}$$

In the second line, we used the fact that $(\operatorname{colim}_{\mathbf{I}}, \Delta)$ form an adjoint pair of functors (the dual result to Theorem 4.6.5). In the last line, we used Theorem 4.6.5. From this, we conclude that

$$Y(X)(\operatorname{colim}_{\mathbf{I}} J) = Hom_{\mathscr{C}}(\operatorname{colim}_{\mathbf{I}} J, X) \cong \varprojlim_{\mathbf{I}} Hom_{\mathscr{C}}(J, X).$$

The functor $Y(X)$ from the Yoneda embedding sends colimits in $\mathscr{C}$ to limits in **Set**. We remark that by a dual argument, the functor $Hom_{\mathscr{C}}(X, -) \in \mathcal{F}(\mathscr{C}, \mathbf{Set})$ preserves limits in $\mathscr{C}$ — it sends a limit in $\mathscr{C}$ to a limit in **Set**.

Now will give a condition necessary for a functor to both preserve and reflect limits. This result explains the name given to the definition below.

Definition 5.1.2. Let $\mathscr{C}, \mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F **creates limits** if the following statement is satisfied: If $\mathbf{I}$ is a small category, $J : \mathbf{I} \to \mathscr{C}$ is a diagram and

$$\big(p_I : L \to (F \circ J)(I)\big)_{I \in \mathbf{I}}$$

is a limit of $F \circ J$ then there exists a limit of J

$$\big(q_I : M \to J(I)\big)_{I \in \mathbf{I}}$$

such that if I is an object in $\mathbf{I}$ then $F(q_I) = p_I$.

Theorem 5.1.2. *Let $\mathscr{C}, \mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Assume that $\boldsymbol{I}$ is a small category and that $\mathscr{D}$ has all limits of shape $\boldsymbol{I}$. Assume that F creates limits. Then limits of shape $\boldsymbol{I}$ exist in $\mathscr{C}$ and F preserves and reflects limits.*

Proof. Assume that $\mathscr{C}, \mathscr{D}$ are categories and $F : \mathscr{C} \to \mathscr{D}$ is a functor. Assume that $\mathbf{I}$ is a small category and $J : \mathbf{I} \to \mathscr{C}$ is a diagram. Firstly, to see that a limit of F exists, we have the limit $\varprojlim_{\mathbf{I}}(F \circ J)$ because $\mathscr{D}$ has all limits of shape $\mathbf{I}$. Since F creates limits then there exists a limit of J

$$\big(q_I : \varprojlim_{\mathbf{I}} F \to J(I)\big)_{I \in \mathbf{I}}$$

such that if I is an object in $\mathbf{I}$ then $(F(q_I))_{I \in \mathbf{I}}$ is the limit of $F \circ J$. Hence $\mathscr{C}$ has all limits of shape $\mathbf{I}$.

Next we will show that F reflects limits. So assume that $(r_I : L \to J(I))_{I \in \mathbf{I}}$ is a cone of J and $(F(r_I) : F(L) \to (F \circ J)(I))_{I \in \mathbf{I}}$ is a limit of $F \circ J$. By

Theorem 4.6.1, $F(L) \cong \varprojlim_{\mathbf{I}}(F \circ J)$ and since F creates limits, $L \cong \varprojlim_{\mathbf{I}} J$ and the cone $(r_I)_{I \in \mathbf{I}}$ is a limit of J. Therefore F reflects limits.

To see that F preserves limits, let $(s_I : M \to J(I))_{I \in \mathbf{I}}$ be a limit of J. We want to show that $(F(s_I) : F(M) \to (F \circ J)(I))_{I \in \mathbf{I}}$ is a limit of $F \circ J$. Again there exists a limit of $F \circ J$ given explicitly by

$$\big(p_I : \varprojlim_{\mathbf{I}}(F \circ J) \to (F \circ J)(I)\big)_{I \in \mathbf{I}}.$$

Since F creates limits then there exists a limit of J

$$\big(t_I : L \to (F \circ J)(I)\big)_{I \in \mathbf{I}}$$

such that $F(t_I) = p_I$. But by uniqueness of the limit in Theorem 4.6.1, $L \cong M$ and subsequently

$$F(M) \cong F(L) = \varprojlim_{\mathbf{I}}(F \circ J).$$

Therefore $(F(s_I))_{I \in \mathbf{I}}$ is a limit of $F \circ J$ and F preserves. $\square$

5.2 Limits and colimits in functor categories

If limits and colimits exist in a functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ then how are they computed? First, we remark that if the categories $\mathscr{C}$ and $\mathscr{D}$ are locally small then it is not true in general that the functor category $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is also locally small. However, if we assume that $\mathscr{C}$ is a small category and $\mathscr{D}$ is locally small then $\mathcal{F}(\mathscr{C}, \mathscr{D})$ is also locally small.

We begin with the following lemma.

Lemma 5.2.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $\boldsymbol{I}$ be a small category. Assume that $\mathscr{D}$ has limits of shape $\boldsymbol{I}$. Let $\lim_{\boldsymbol{I}} : \mathcal{F}(\boldsymbol{I}, \mathscr{D}) \to \mathscr{D}$ be the limit functor in Definition 4.6.1 and $\Delta : \mathscr{D} \to \mathcal{F}(\boldsymbol{I}, \mathscr{D})$ be the diagonal functor in Definition 4.6.2. Define the functors*

$$\begin{array}{rccc} \Psi : & \mathcal{F}(\mathscr{C}, \mathscr{D}) & \to & \mathcal{F}(\mathscr{C}, \mathcal{F}(\boldsymbol{I}, \mathscr{D})) \\ & H & \mapsto & \Delta \circ H \\ & \xi : H_1 \Rightarrow H_2 & \mapsto & \big(\Delta(\xi_C)\big)_{C \in \mathscr{C}} \end{array}$$

and

$$\begin{array}{rccc}\Phi: & \mathcal{F}(\mathscr{C},\mathcal{F}(\boldsymbol{I},\mathscr{D})) & \to & \mathcal{F}(\mathscr{C},\mathscr{D}) \\ & G & \mapsto & \lim_{\boldsymbol{I}} \circ G \\ & \eta: G_1 \Rightarrow G_2 & \mapsto & \big(\lim_{\boldsymbol{I}}(\eta_C)\big)_{C\in\mathscr{C}}.\end{array}$$

Then (Ψ, Φ) *is an adjoint pair of functors.*

Proof. Assume that Φ and Ψ are the functors defined in the statement of the lemma. We will use Theorem 2.2.2 to show that Φ and Ψ define an adjoint pair of functors.

We know from Theorem 4.6.5 that the pair $(\Delta, \lim_{\mathbf{I}})$ is an adjoint pair of functors. By Theorem 2.2.1, we obtain the unit and counit of adjunction

$$\eta : id_{\mathscr{D}} \Rightarrow \lim_{\mathbf{I}} \circ \Delta \qquad \text{and} \qquad \epsilon : \Delta \circ \lim_{\mathbf{I}} \Rightarrow id_{\mathcal{F}(\mathbf{I},\mathscr{D})}$$

respectively. Now define

$$\mu : id_{\mathcal{F}(\mathscr{C},\mathscr{D})} \Rightarrow \Phi \circ \Psi \qquad \text{and} \qquad \nu : \Psi \circ \Phi \Rightarrow id_{\mathcal{F}(\mathscr{C},\mathcal{F}(\mathbf{I},\mathscr{D}))}$$

in the following manner: If $F \in \mathcal{F}(\mathscr{C},\mathscr{D})$ then $\mu_F : F \Rightarrow (\Phi \circ \Psi)(F)$ is the natural transformation given by the family of morphisms

$$\big(\mu_{F,C} = \eta_{F(C)} : F(C) \Rightarrow (\lim_{\mathbf{I}} \circ \Delta)(F(C))\big)_{C\in\mathscr{C}}$$

If $K \in \mathcal{F}(\mathscr{C},\mathcal{F}(\mathbf{I},\mathscr{D}))$ then $\nu_K : (\Psi \circ \Phi)(K) \Rightarrow K$ is the natural transformation given by the family of morphisms

$$\big(\nu_{K,C} = \epsilon_{K(C)} : (\Delta \circ \lim_{\mathbf{I}})(K(C)) \Rightarrow K(C)\big)_{C\in\mathscr{C}}.$$

It is a tedious exercise to check that μ and ν are natural transformations. Now if $H \in \mathcal{F}(\mathscr{C},\mathcal{F}(\mathbf{I},\mathscr{D}))$ and $C \in \mathscr{C}$ then by Theorem 2.2.1,

$$\begin{aligned}(\Phi(\nu_H) \circ \mu_{\Phi(H)})_C &= (\Phi(\nu_H))_C \circ \mu_{\Phi(H),C} \\ &= (\Phi(\nu_H))_C \circ \eta_{\Phi(H)(C)} \\ &= \lim_{\mathbf{I}}(\nu_{H,C}) \circ \eta_{\Phi(H)(C)} \\ &= \lim_{\mathbf{I}}(\epsilon_{H(C)}) \circ \eta_{\lim_{\mathbf{I}} H(C)} \\ &= id_{\lim_{\mathbf{I}} H(C)}.\end{aligned}$$

Since C is an arbitrary object in $\mathscr{C}$ then

$$\Phi(\nu_H) \circ \mu_{\Phi(H)} = id_{\lim_{\mathbf{I}} \circ H} = id_{\Phi(H)}.$$

Also if $F \in \mathcal{F}(\mathscr{C}, \mathscr{D})$ and $C \in \mathscr{C}$ then by Theorem 2.2.1,

$$\begin{aligned}(\nu_{\Psi(F)} \circ \Psi(\mu_F))_C &= \nu_{\Psi(F),C} \circ (\Psi(\mu_F))_C \\ &= \epsilon_{\Psi(F)(C)} \circ \Delta(\mu_{F,C}) \\ &= \epsilon_{\Delta(F(C))} \circ \Delta(\eta_{F(C)}) \\ &= id_{\Delta(F(C))}.\end{aligned}$$

Again since C is an arbitrary object in $\mathscr{C}$,

$$\nu_{\Psi(F)} \circ \Psi(\mu_F) = id_{\Delta \circ F} = id_{\Psi(F)}.$$

By Theorem 2.2.2, the pair (Ψ, Φ) is an adjoint pair of functors. □

Now we will describe limits in a functor category.

Theorem 5.2.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $\boldsymbol{I}$ be a small category. Assume that $\mathscr{D}$ has limits of shape $\boldsymbol{I}$. Let Φ be the right adjoint functor in Lemma 5.2.1. The composite of functors*

$$\mathcal{F}(\boldsymbol{I}, \mathcal{F}(\mathscr{C}, \mathscr{D})) \xrightarrow{\cong} \mathcal{F}(\mathscr{C}, \mathcal{F}(\boldsymbol{I}, \mathscr{D})) \xrightarrow{\Phi} \mathcal{F}(\mathscr{C}, \mathscr{D}) \tag{5.1}$$

is the limit functor $\lim_{\boldsymbol{I}}^{\mathcal{F}(\mathscr{C},\mathscr{D})} : \mathcal{F}(\boldsymbol{I}, \mathcal{F}(\mathscr{C}, \mathscr{D})) \to \mathcal{F}(\mathscr{C}, \mathscr{D})$. Dually, if $\mathscr{D}$ has colimits of shape $\boldsymbol{I}$ then $\mathcal{F}(\mathscr{C}, \mathscr{D})$ has colimits and hence a colimit functor.

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories. Assume that $\mathbf{I}$ is a small category and $\mathscr{D}$ has limits of shape $\mathbf{I}$. By Lemma 5.2.1, the pair of functors (Ψ, Φ) is an adjoint pair of functors. Let

$$\tau : \mathcal{F}(\mathscr{C}, \mathcal{F}(\mathbf{I}, \mathscr{D})) \to \mathcal{F}(\mathbf{I}, \mathcal{F}(\mathscr{C}, \mathscr{D}))$$

denote the isomorphism. The composite $\tau \circ \Psi$ sends a functor $F : \mathscr{C} \to \mathscr{D}$ to the functor

$$\begin{array}{rcl}\mathbf{I} & \to & \mathcal{F}(\mathscr{C}, \mathscr{D}) \\ i & \mapsto & \big(C \mapsto (\Delta \circ F)(C)(i)\big)\end{array}$$

But $(\Delta \circ F)(C)(i) = F(C)$ by definition of the diagonal functor Δ in Definition 4.6.2. We conclude that the composite $\tau \circ \Psi$ is the diagonal functor

$$\Delta^{\mathcal{F}(\mathscr{C},\mathscr{D})} : \mathcal{F}(\mathscr{C}, \mathscr{D}) \to \mathcal{F}(\mathbf{I}, \mathcal{F}(\mathscr{C}, \mathscr{D}))$$

Since (Ψ, Φ) is an adjoint pair of functors then the associated right adjoint functor to $\Delta^{\mathcal{F}(\mathscr{C},\mathscr{D})}$ is

$$\Phi \circ \tau^{-1} : \mathcal{F}(\mathbf{I}, \mathcal{F}(\mathscr{C}, \mathscr{D})) \to \mathcal{F}(\mathscr{C}, \mathscr{D})$$

which is the composite in equation (5.1). By Theorem 4.6.5, $\Phi \circ \tau^{-1}$ is the limit functor $\lim_{\mathbf{I}}^{\mathcal{F}(\mathscr{C},\mathscr{D})} : \mathcal{F}(\mathbf{I}, \mathcal{F}(\mathscr{C}, \mathscr{D})) \to \mathcal{F}(\mathscr{C}, \mathscr{D})$.

The dual statement for colimits is proved in a similar manner. □

Example 5.2.1. The construction in Theorem 5.2.2 tells us that limits in functor categories are computed pointwise. Here is a significant example to demonstrate this. Let $\mathscr{C}$ be a small category. The category **Set** is finitely complete and finitely cocomplete. By Theorem 5.2.2, the category of (set-valued) presheaves $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ must also admit small limits and colimits.

Now recall the Yoneda embedding $Y : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ from Definition 3.1.2. Suppose that $\mathscr{C}$ is a complete category and let L be a limit in $\mathscr{C}$. Then L is a colimit in $\mathscr{C}^{op}$. By Example 5.1.1, $Y(L)$ is a limit in **Set**. Therefore the Yoneda embedding preserves limits.

On the contrary, we do not expect the Yoneda embedding to preserve colimits. Let us now analyse how it interacts with colimits. Recall the definition of a comma category from Definition 2.3.2. Let $\mathscr{C}$ be a category and $F : \mathscr{C}^{op} \to \mathbf{Set}$ be a presheaf. Let X be an object in $\mathscr{C}$, $\mathbf{1}$ be the category with the single object $\bullet$ and $\mathbb{1}_{\{*\}} : \mathbf{1} \to \mathbf{Set}$ be the functor in equation (2.6). The category $(\mathbb{1}_{\{*\}}, F)^{op}$ has objects given by triples (A, f, B) where $A \in \mathbf{1}$, $B \in \mathscr{C}$ and $f \in Hom_{\mathbf{Set}}(\{*\}, F(B))$. Hence an object (A, f, B) is equivalent to a pair (B, b) where $B \in \mathscr{C}$ and $b \in F(B)$.

In order to describe the morphisms in $(\mathbb{1}_{\{*\}}, F)^{op}$, let us first describe them in the comma category $(\mathbb{1}_{\{*\}}, F)$. A morphism $\alpha : (A, f, B) \to (P, g, Q)$ in $(\mathbb{1}_{\{*\}}, F)$ is a pair (β, γ) where $\beta \in Hom_{\mathbf{1}}(A, P)$, $\gamma \in Hom_{\mathscr{C}}(B, Q)$ and $g \circ \mathbb{1}_{\{*\}}(\beta) = F(\gamma) \circ f$. This is equivalent to a morphism $\gamma : B \to Q$ in $\mathscr{C}$ which induces a function $F(\gamma) : F(B) \to F(Q)$ which sends the specific element $f(*) \in F(B)$ to the specific element $g(*) \in F(Q)$.

Not only is $F(\gamma)$ a morphism of sets, it also maps a specific element to another specific element. The way we make sense of this is through the category of *pointed sets*.

Definition 5.2.1. The category of **pointed sets**, denoted by $\mathbf{Set}_*$, is the category whose objects are pairs (X, x) consisting of a set X and a point

$x \in X$. A morphism $\phi : (X, x) \to (Y, y)$ in $\mathbf{Set}_*$ is simply a morphism of sets such that $\phi(x) = y$.

Therefore a morphism $\alpha : (A, f, B) \to (P, g, Q)$ in the opposite category $(\mathbb{1}_{\{*\}}, F)^{op}$ can be thought of as a morphism $\gamma : B \to Q$ in $\mathscr{C}$ which induces a morphism of pointed sets $F(\gamma) : (F(Q), g(*)) \to (F(B), f(*))$. In the statement of the following lemma, we will implicitly make use of the fact that if $\mathscr{C}$ is a small category then $(\mathbb{1}_{\{*\}}, F)^{op}$ is also small.

Lemma 5.2.3. *Let $\mathscr{C}$ be a small category. Let $F, G \in \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set})$ be presheaves. Let $Y : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set})$ be the Yoneda embedding in Definition 3.1.2,*

$$\Delta : \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}) \to \mathcal{F}((\mathbb{1}_{\{*\}}, F)^{op}, \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}))$$

be the diagonal functor (see Definition 4.6.2) and P be the functor

$$\begin{array}{rccc} P : & (\mathbb{1}_{\{*\}}, F)^{op} & \to & \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}) \\ & (X \in \mathscr{C}, x \in F(X)) & \mapsto & Y(X). \end{array}$$

Then there is a natural isomorphism

$$\tau_{F,G} : Hom_{\mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set})}(F, G) \to Hom_{\mathcal{F}((\mathbb{1}_{\{*\}}, F)^{op}, \mathcal{F}(\mathscr{C}^{op}, \boldsymbol{Set}))}(P, \Delta(G)).$$

Proof. Assume that $\mathscr{C}$ is a small category and $F, G \in \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ are presheaves. Assume that Δ and P are the functors defined as above. Let $\eta \in Hom_{\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})}(F, G)$. Then η is a natural transformation from F to G. If (X, y) is an object in the category $(\mathbb{1}_*, F)^{op}$ and X' is an object in $\mathscr{C}$ then define the morphism

$$\begin{array}{rccc} \zeta_{(X,y),X'} : & P(X, y)(X') = Hom_{\mathscr{C}}(X', X) & \to & \Delta(G)(X, y)(X') = G(X') \\ & g & \mapsto & G(g)(\eta_X(y)). \end{array}$$

The family $(\zeta_{(X,y),X'})_{X' \in \mathscr{C}}$ yields a natural transformation $\zeta_{(X,y)} : P(X, y) \Rightarrow \Delta(G)(X, y)$. One can also check that this gives a natural transformation $\zeta : P \Rightarrow \Delta(G)$ and hence an element of the class

$$Hom_{\mathcal{F}((\mathbb{1}_{\{*\}}, F)^{op}, \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}))}(P, \Delta(G)).$$

The construction of ζ from η gives us a map

$$\tau_{F,G} : Hom_{\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})}(F, G) \to Hom_{\mathcal{F}((\mathbb{1}_{\{*\}}, F)^{op}, \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}))}(P, \Delta(G)).$$

To see that $\tau_{F,G}$ is bijective, assume that $\zeta : P \Rightarrow \Delta(G)$ is a natural transformation. We will show that there exists a unique $\eta : F \Rightarrow G$ such that $\tau_{F,G}(\eta) = \zeta$.

Assume that $\gamma : (X, y) \to (X', y')$ is a morphism in $(\mathbb{1}_{\{*\}}, F)^{op}$. If Z is an object in $\mathscr{C}$ then the following diagram in **Set** commutes

$$\begin{array}{ccc} P(X,y)(Z) & \xrightarrow{P(\gamma)(Z)} & P(X',y')(Z) \\ \downarrow{\scriptstyle \zeta_{(X,y),Z}} & & \downarrow{\scriptstyle \zeta_{(X',y'),Z}} \\ \Delta(X,y)(Z) & \xrightarrow[\Delta(G)(\gamma)(Z)]{} & \Delta(X',y')(Z) \end{array}$$

because ζ is a natural transformation. By definition of the functors P and Δ, the above diagram simplifies to

$$\begin{array}{ccc} Hom_{\mathscr{C}}(Z,X) & \xrightarrow{Y(\gamma)=\gamma_*} & Hom_{\mathscr{C}}(Z,X') \\ \downarrow{\scriptstyle \zeta_{(X,y),Z}} & & \downarrow{\scriptstyle \zeta_{(X',y'),Z}} \\ G(Z) & \xrightarrow[id_Z]{} & G(Z) \end{array}$$

where we recall that γ_* denotes composition with γ. In the above diagram, if we set $Z = X$ then commutativity gives us

$$\zeta_{(X,y),X}(id_X) = (\zeta_{(X',y'),Z} \circ \gamma^*)(id_X) = \zeta_{(X',y'),X}(\gamma).$$

In particular, the above equation holds in the set $G(X)$. Hence if X is an object in $\mathscr{C}$ then define the functions

$$\begin{array}{rccc} \eta_X : & F(X) & \to & G(X) \\ & y & \mapsto & \zeta_{(X,y),X}(id_X). \end{array}$$

To see that $(\eta_X)_{X \in \mathscr{C}}$ defines a natural transformation from F to G, let $\phi : A \to B$ be a morphism in $\mathscr{C}^{op}$. If $b \in F(B)$ then by thinking of $F(\phi)$ as a morphism of pointed sets from $(F(B), b)$ to $(F(A), F(\phi)(b))$, we can think of ϕ as a morphism in $(\mathbb{1}_*, F)^{op}$ from $(A, F(\phi)(b))$ to (B, b). So

$$\begin{aligned} (\eta_A \circ F(\phi))(b) &= \eta_A(F(\phi)(b)) \\ &= \zeta_{(A,F(\phi)(b)),A}(id_A) \\ &= \zeta_{(B,b),A}(\phi) \\ &= G(\phi) \circ \zeta_{(B,b),B}(id_B) \\ &= (G(\phi) \circ \eta_B)(b). \end{aligned}$$

In the second last line, we used the fact that $\zeta_{(B,b)}$ is itself a natural transformation from $P(B,b) = Y(B)$ to $\Delta(G)(B,b) = G$. We conclude that

η is a natural transformation from F to G.

Now if $g \in Hom_{\mathscr{C}}(X', X)$ and $(X, y) \in (\mathbb{1}_*, F)^{op}$ then

$$\begin{aligned}\zeta_{(X,y),X'}(g) &= \zeta_{(X',F(g)(y)),X'}(id_{X'}) \\ &= \eta_{X'}(F(g)(y)) \\ &= (\eta_{X'} \circ F(g))(y) = G(g)(\eta_X(y)).\end{aligned}$$

Since g and (X, y) were arbitrary then $\tau_{F,G}(\eta) = \zeta$. It is also straightforward to check that η is unique because η_X must send y to $\zeta_{(X,y),X}(id_X)$ for the entire construction to work. This shows that $\tau_{F,G}$ is a bijection. We omit the plethora of computations required to show that $\tau_{F,G}$ is natural in both F and G. □

We know that colimit functors are left adjoint to diagonal functors. Hence, we have the following corollary of Lemma 5.2.3.

Corollary 5.2.4. *Let $\mathscr{C}$ be a small category. Let $F \in \mathcal{F}(\mathscr{C}^{op}, \textbf{Set})$ be a presheaf. Let $Y : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \textbf{Set})$ be the Yoneda embedding in Definition 3.1.2,*

$$\Delta : \mathcal{F}(\mathscr{C}^{op}, \textbf{Set}) \to \mathcal{F}((\mathbb{1}_{\{*\}}, F)^{op}, \mathcal{F}(\mathscr{C}^{op}, \textbf{Set}))$$

be the diagonal functor (see Definition 4.6.2) and P be the functor

$$\begin{array}{cccc} P: & (\mathbb{1}_{\{*\}}, F)^{op} & \to & \mathcal{F}(\mathscr{C}^{op}, \textbf{Set}) \\ & (X \in \mathscr{C}, x \in F(X)) & \mapsto & Y(X). \end{array}$$

Then we have an isomorphism

$$F \cong colim_{(\mathbb{1}_{\{*\}}, F)^{op}} P.$$

Corollary 5.2.4 tells us that any presheaf is a colimit of representable functors. We say that the functor category of presheaves $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ is the **free cocompletion** of $\mathscr{C}$.

Corollary 5.2.5. *Let $\mathscr{C}$ be a small category and $\mathscr{D}$ be a category which has small colimits. Let $\mathcal{F}^{colim}(\mathcal{F}(\mathscr{C}^{op}, \textbf{Set}), \mathscr{D})$ be the full subcategory of functors which preserve small colimits. Then there is an equivalence of categories*

$$\mathcal{F}^{colim}(\mathcal{F}(\mathscr{C}^{op}, \textbf{Set}), \mathscr{D}) \cong \mathcal{F}(\mathscr{C}, \mathscr{D}).$$

Proof. Assume that $\mathscr{C}$ is a small category and $\mathscr{D}$ be a category which has small colimits. Let $F : \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \to \mathscr{D}$ be a functor. By restricting F to representable presheaves (see Definition 3.2.1), we obtain a functor from $\mathscr{C}$ to $\mathscr{D}$.

On the other hand, let $G : \mathscr{C} \to \mathscr{D}$ be a functor. We will define a small colimit-preserving functor $\widehat{G} \in \mathcal{F}^{\text{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D})$. If H is a presheaf then by Corollary 5.2.4,

$$H \cong \text{colim}_{(\mathbb{1}_{\{*\}}, H)^{op}} P$$

where P is the functor in Lemma 5.2.3. Now define $\widehat{G}$ to be the functor

$$\begin{array}{rccc} \widehat{G}: & \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) & \to & \mathscr{D} \\ & \text{colim}_{(\mathbb{1}_{\{*\}}, H)^{op}} P & \mapsto & \text{colim}_{(\mathbb{1}_{\{*\}}, H)^{op}} (G \circ P). \end{array}$$

We remark here that the image of the functor P is the set of representable presheaves in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$. This means that the composite $G \circ P$ makes sense.

Now we will show that $\widehat{G}$ preserves small colimits. Let $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ be a diagram of presheaves. By definition of $\widehat{G}$,

$$\widehat{G}(\text{colim}_{\mathbf{I}} D) = \text{colim}_{(\mathbb{1}_{\{*\}}, \text{colim}_{\mathbf{I}} D)^{op}} (G \circ P).$$

But if I is an object in $\mathbf{I}$ then we also have

$$\text{colim}_{\mathbf{I}} \widehat{G}(D_I) = \text{colim}_{\mathbf{I}}\ \text{colim}_{(\mathbb{1}_{\{*\}}, D_I)^{op}} (G \circ P)$$

Both of these colimits are isomorphic. So, $\widehat{G}$ preserves small colimits. Now suppose that $H \in \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ is a representable presheaf. Then there exists an object X in $\mathscr{C}$ such that $Y(X) \cong H$. So

$$\widehat{G}(H) = \widehat{G}(Y(X)) = \text{colim}_{(\mathbb{1}_{\{*\}}, Y(X))^{op}} (G \circ P).$$

The category $(\mathbb{1}_{\{*\}}, Y(X))^{op}$ has a terminal object given by the pair (X, id_X). Consequently, $\widehat{G}(Y(X)) \cong G(X)$. This means that if $\nu : \mathcal{F}(\mathscr{C}, \mathscr{D}) \to \mathcal{F}^{\text{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D})$ is the map $G \mapsto \widehat{G}$ and $\rho : \mathcal{F}^{\text{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D}) \to \mathcal{F}(\mathscr{C}, \mathscr{D})$ is restriction to representable presheaves then the composite

$$\mathcal{F}(\mathscr{C}, \mathscr{D}) \xrightarrow{\nu} \mathcal{F}^{\text{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D}) \xrightarrow{\rho} \mathcal{F}(\mathscr{C}, \mathscr{D})$$

is naturally isomorphic to the identity.

Now assume that $F \in \mathcal{F}^{\mathrm{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D})$. By Corollary 5.2.4, F is *uniquely* determined on the subcategory of representable presheaves in $\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ because every presheaf is a colimit of representable presheaves. So the composite

$$\mathcal{F}^{\mathrm{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D}) \xrightarrow{\rho} \mathcal{F}(\mathscr{C}, \mathscr{D}) \xrightarrow{\nu} \mathcal{F}^{\mathrm{colim}}(\mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}), \mathscr{D})$$

is naturally isomorphic to the identity. This completes the proof. □

5.3 Adjoint functor theorem

Recall from Theorem 5.1.1 that if $F : \mathscr{C} \to \mathscr{D}$ is a left adjoint functor then it must preserve colimits. Is there a converse to this statement? In general, even if $\mathscr{C}$ has small limits and F preserves colimits, a right adjoint to F may not exist. The point of the *adjoint functor theorem* is that it supplies the extra conditions on F required for it to admit a right adjoint (and hence be a left adjoint functor).

First, we remind the reader of Definition 3.2.2 and Theorem 3.2.2, which gives us an equivalent condition for $F : \mathscr{C} \to \mathscr{D}$ to have a right adjoint. Succinctly, F has a right adjoint functor if its formal right adjoint $G^{form} : \mathscr{D} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ can be lifted along the Yoneda embedding.

Definition 5.3.1. Let $\mathscr{C}$ be a category and Y be the Yoneda embedding in equation (3.1). We say that $\mathscr{C}$ is **total** if Y has a left adjoint, which we denote by $Y^L : \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \to \mathscr{C}$.

Now if $Q \in \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ is a presheaf and P is the functor in Corollary 5.2.4 then by Corollary 5.2.4 and Theorem 5.1.1,

$$Y^L(Q) \cong Y^L(\mathrm{colim}_{(\mathbb{1}_{\{*\}}, Q)^{op}} P) = \mathrm{colim}_{x \in \mathscr{C},\, z \in P(x)} Y^L(Y(x)) \cong \mathrm{colim}_{x \in \mathscr{C},\, z \in P(x)} x.$$

Hence, a category $\mathscr{C}$ is total if the above colimit exists.

Example 5.3.1. Let k be a field. The categories **Set** and k-**Vect** are both total categories.

Theorem 5.3.1 (Adjoint functor theorem V1)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Assume that $\mathscr{C}$ is total. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F preserves colimits if and only if F has a right adjoint.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are locally small categories and that $\mathscr{C}$ is total. Assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor. We know from Theorem 5.1.1 that if F is a left adjoint functor then it preserves colimits.

Conversely, assume that F preserves colimits. Let $G^{form} : \mathscr{D} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ be the formal right adjoint of F (see Definition 3.2.2). Since $\mathscr{C}$ is total then it has a left adjoint $Y^L : \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set}) \to \mathscr{C}$. Define the functor

$$G = Y^L \circ G^{form}.$$

What does G do explicitly? If y is an object in $\mathscr{D}$ then

$$Gy = (Y^L \circ G^{form})(y) = Y^L(G^{form}(y)) = \mathrm{colim}_{x\in\mathscr{C},\ z\in Hom_{\mathscr{D}}(Fx,y)} x.$$

The colimit $\mathrm{colim}_{x\in\mathscr{C},\ z\in Hom_{\mathscr{D}}(Fx,y)} x$ is explicitly a family of morphisms in $\mathscr{C}$ given by

$$\big(\chi_{(x,z)} : x \to Gy\big)_{(x,z)\in(\mathbb{1}_{\{*\}}, G^{form}(y))^{op}}$$

Hence if x is an object in $\mathscr{C}$ and y is an object in $\mathscr{D}$ then we have a family of maps

$$\begin{array}{rccc} \psi_{(x,y)}: & Hom_{\mathscr{D}}(Fx,y) & \to & Hom_{\mathscr{C}}(x,Gy) \\ & z & \mapsto & \chi_{(x,z)}. \end{array}$$

To show: (a) $\psi_{(x,y)}$ is natural in both x and y.

(a) Assume that $f : x_1 \to x_2$ is a morphism in $\mathscr{C}$ and y is an object in $\mathscr{D}$. If $z \in Hom_{\mathscr{D}}(Fx_2, y)$ then

$$\begin{aligned} (\psi_{(x_1,y)} \circ F(f)_*)(z) &= \psi_{(x_1,y)}(z \circ F(f)) \\ &= \chi_{(x_1, z\circ F(f))} \\ &= \chi_{(x_2,z)} \circ f \\ &= f_*(\chi_{(x_2,z)}) \\ &= (f_* \circ \psi_{(x_2,y)})(z). \end{aligned}$$

In the third line, we used the universal property of the colimit. Hence $\psi_{(x,y)}$ is natural in x. Now assume that $g : y_1 \to y_2$ is a morphism in $\mathscr{D}$. If $z \in Hom_{\mathscr{D}}(Fx, y_1)$ then

$$\begin{aligned}(G(g)^* \circ \psi_{(x,y_1)})(z) &= G(g)^*(\chi_{(x,z)}) \\ &= G(g) \circ \chi_{(x,z)} \\ &= \chi_{(x,g\circ z)} = \psi_{(x,y_2)}(g \circ z) \\ &= (\psi_{(x,y_2)} \circ g^*)(z)\end{aligned}$$

So $\psi_{(x,y)}$ is natural in y.

By repeating the proof of Theorem 2.2.1, we use the map $\psi_{(x,y)}$ to construct a unit of adjunction $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$. Next, we want to construct a counit of adjunction. By definition of the counit, it suffices to show that if $y \in \mathscr{D}$ then there exists $t \in Hom_{\mathscr{D}}(FGy, y)$ such that $\psi_{(x,y)}(t) = id_{Gy}$.

Observe that

$$\begin{aligned}Hom_{\mathscr{D}}(FGy, y) &\cong Hom_{\mathscr{D}}(F(\mathrm{colim}_{(x,z)} x), y) \\ &\cong Hom_{\mathscr{D}}(\mathrm{colim}_{(x,z)} F(x), y) \\ &\cong \lim_{x\in\mathscr{C},\, z\in Hom(Fx,y)} Hom_{\mathscr{D}}(Fx, y).\end{aligned}$$

In the second line, we used the assumption that F preserves colimits. In the final line, we used the fact that the functor $Y(y) = Hom_{\mathscr{D}}(-, y)$ maps colimits in $\mathscr{D}$ to limits in **Set**.

Now if $(x, z) \in (\mathbb{1}_{\{*\}}, G^{form}(y))^{op}$ then we have a family of morphisms

$$\begin{array}{rccc}\sigma_{(x,z)} : & Hom_{\mathscr{D}}(Fx, y) & \to & Hom_{\mathscr{D}}(Fx, y) = G^{form}(y)(x) \\ & r & \mapsto & z.\end{array}$$

Now consider the composite

$$\lim_{(x,z)} Hom_{\mathscr{D}}(Fx, y) \longrightarrow \lim_{(x,z)} Hom_{\mathscr{D}}(x, Gy) \xrightarrow{\cong} Hom_{\mathscr{C}}(Gy, Gy).$$

One can check that this composite sends $(\sigma_{(x,z)})_{(x,z)}$ to id_{Gy}. So by the isomorphism $Hom_{\mathscr{D}}(FGy, y) \cong \lim_{(x,z)} Hom_{\mathscr{D}}(Fx, y)$, there exists $t \in Hom_{\mathscr{D}}(FGy, y)$ such that $\psi_{(x,y)}(t) = id_{Gy}$. By Theorem 2.2.2, (F, G) is an adjoint pair of functors. □

The main issue with the formulation of the adjoint functor theorem in Theorem 5.3.1 is that generally, it is not easy to check if a given category is total. Our next formulation of the adjoint functor theorem uses the more palatable definition of a *locally presentable category*.

Definition 5.3.2. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is **essentially small** if there exists a small category $\mathscr{D}$ and an equivalence $F : \mathscr{C} \to \mathscr{D}$.

For the next definition, if $\mathscr{D}$ is a category then we let $\mathcal{F}_{lex}(\mathscr{D}, \mathbf{Set})$ be the subcategory of $\mathcal{F}(\mathscr{D}, \mathbf{Set})$ of functor which preserve finite limits.

Definition 5.3.3. Let $\mathscr{C}$ be a locally small category. We say that $\mathscr{C}$ is **locally presentable** if there exists an essentially small subcategory $\mathscr{C}^c \hookrightarrow \mathscr{C}$ of *compact objects* such that if $Y : \mathscr{C} \to \mathcal{F}(\mathscr{C}^{op}, \mathbf{Set})$ is the Yoneda embedding in equation (3.1) then its restriction

$$Y|_{\mathcal{F}_{lex}((\mathscr{C}^c)^{op}, \mathbf{Set})} : \mathscr{C} \to \mathcal{F}_{lex}((\mathscr{C}^c)^{op}, \mathbf{Set})$$

is an equivalence.

Example 5.3.2. We will now state some well-known examples of locally presentable categories. The category **Set** is locally presentable; the compact objects in **Set** are the finite sets. Similarly, if k is a field then k-**Vect** is locally presentable with its compact objects being finite dimensional k-vector spaces. The category **Grp** is locally presentable with its compact objects being finitely presented groups.

In [ABLR02, Theorem 5.5], we have the following alternative characterisation of a locally presentable category.

Theorem 5.3.2. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is locally presentable if and only if the following statement is satisfied: there exists an essentially small category $\mathscr{D}$ and a fully faithful functor $i : \mathscr{C} \hookrightarrow \mathcal{F}(\mathscr{D}^{op}, \mathbf{Set})$ such that i has a left adjoint and commutes with filtered colimits.*

A particular consequence of Theorem 5.3.2 is that if $\mathscr{C}$ is locally presentable then the fully faithful functor i in the statement of Theorem 5.3.2 is just the embedding $\mathcal{F}_{lex}((\mathscr{C}^c)^{op}, \mathbf{Set}) \hookrightarrow \mathcal{F}((\mathscr{C}^c)^{op}, \mathbf{Set})$. Theorem 5.3.2 tells us that the category of presheaves $\mathcal{F}((\mathscr{C}^c)^{op}, \mathbf{Set})$ is obtained from $\mathscr{C} \cong \mathcal{F}_{lex}((\mathscr{C}^c)^{op}, \mathbf{Set})$ by freely adding colimits of compact objects.

Now we state and prove our second variant of the adjoint functor theorem.

Theorem 5.3.3 (Adjoint functor theorem V2)**.** *Let $\mathscr{C}$ be a locally presentable category and $\mathscr{D}$ be a locally small category. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F has a right adjoint if and only if it preserves colimits.*

Proof. Assume that $\mathscr{C}$ is a locally presentable category and $\mathscr{D}$ is a locally small category. Assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor. We know from

Theorem 5.1.1 that if F is a left adjoint functor then it preserves colimits.

To show: (a) If F preserves colimits then F has a right adjoint.

(a) By assumption, $\mathscr{C}$ is a locally presentable category. So let $\mathscr{C}^c$ denote the subcategory of compact objects in $\mathscr{C}$. Then we have a functor

$$\begin{array}{rrcl} G: & \mathscr{D} & \to & \mathcal{F}((\mathscr{C}^c)^{op}, \mathbf{Set}) \\ & Y & \mapsto & \big(X \mapsto Hom_{\mathscr{D}}(F(X), Y)\big). \end{array}$$

If Y is an object in $\mathscr{D}$ then the functor $G(Y)$ preserves finite limits because F preserves colimits in $\mathscr{C}$ (which are limits in the opposite category). Therefore the image of G is the subcategory $\mathcal{F}_{lex}((\mathscr{C}^c)^{op}, \mathbf{Set})$ which is equivalent to $\mathscr{C}$ by the definition of local presentability.

By Lemma 3.1.1, if X is an object in $\mathscr{C}^c$, Z is an object in $\mathscr{D}$ and Y is the Yoneda embedding in equation (3.1) then

$$Hom_{\mathscr{D}}(F(X), Z) = G(Z)(X) \cong Y|_{\mathcal{F}_{lex}((\mathscr{C}^c)^{op}, \mathbf{Set})}(G(Z))(X) = Hom_{\mathscr{C}}(X, G(Z)).$$

To see that the above bijection holds for any $X \in \mathscr{C}$, first note that if X is an object in $\mathscr{C}$ then it is a colimit of compact objects because $\mathscr{C}$ is locally presentable. Now F preserves colimits and the presheaf $G(Z)$ is a colimit of representable functors by Corollary 5.2.4, which also preserve colimits in $\mathscr{C}$ (see Example 5.1.1). Together, this demonstrates that the bijection $Hom_{\mathscr{C}}(F(X), Z) \cong Hom_{\mathscr{D}}(X, G(Z))$ holds for arbitrary objects $X \in \mathscr{C}$ and $Z \in \mathscr{D}$ as required. □

The most general form of the adjoint functor theorem is attributed to Freyd and makes use of Theorem 2.3.2. Recall that if $F : \mathscr{C} \to \mathscr{D}$ is a functor and $\mathbf{1}$ is the category with a single object $*$ then F has a right adjoint functor if and only if the following statement is satisfied: If X is an object in $\mathscr{D}$ and $\mathbb{1}_X$ is the functor defined in equation (2.6) then the comma category $(F, \mathbb{1}_X)$ has a terminal object, which we know is a limit of the empty diagram.

However, we can also express this terminal object as a colimit over the identity functor. This colimit is not small unless the category itself is small. The point here is that we can construct this terminal object of $(F, \mathbb{1}_X)$ if we reduce the colimit to a small colimit. The way this is done is to play around with the set-theoretic issues which arise. This leads to the definition of the solution set condition.

Definition 5.3.4. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ satisfies the **solution set condition** if there exist a (small) set I and a family of objects $\{c_i\}_{i\in I}$ in $\mathscr{C}$ such that if $x \in \mathscr{C}$ then there exists $i \in I$ such that $Hom_{\mathscr{C}}(x, c_i) \neq \emptyset$.

Theorem 5.3.4. *Let $\mathscr{C}$ be a locally small category with small colimits (a locally small, cocomplete category). Then $\mathscr{C}$ has a terminal object if and only if it satisfies the solution set condition.*

Proof. Assume that $\mathscr{C}$ is a locally small category with small colimits.

To show: (a) If $\mathscr{C}$ has a terminal object then $\mathscr{C}$ satisfies the solution set condition.

(b) If $\mathscr{C}$ satisfies the solution set condition then it has a terminal object.

(a) Assume that $*$ is a terminal object in $\mathscr{C}$. If X is an object in $\mathscr{C}$ then $Hom_{\mathscr{C}}(X, *) \neq \emptyset$ because it contains the terminal map from X to $*$. The set $\{*\}$, together with the family of objects $\{*\}$ in $\mathscr{C}$, demonstrates that $\mathscr{C}$ satisfies the solution set condition.

(b) Assume that $\mathscr{C}$ satisfies the solution set condition with the set I and the family of objects $\{C_i\}_{i\in I}$ in $\mathscr{C}$. Since $\mathscr{C}$ is cocomplete, let W be the coproduct

$$W = \coprod_{i\in I} C_i.$$

Since $\mathscr{C}$ is a locally small category, the class $Hom_{\mathscr{C}}(W, W)$ is actually a set. Now let $\mathbf{W}$ be the small category with one object and the same number of morphisms (on the single object) as elements in $Hom_{\mathscr{C}}(W, W)$. We have a diagram $D : \mathbf{W} \to \mathscr{C}$ which maps the single object in $\mathbf{W}$ to W. Let

$$C = \mathrm{colim}_{\mathbf{W}} D$$

The object C in $\mathscr{C}$ can be thought of as the coequalizer of all the morphisms from W to W. We claim that C is in fact the terminal object in $\mathscr{C}$.

First we make use of the fact that W is a coproduct and C is a "coequalizer". Let X be an object in $\mathscr{C}$. Then there exists $i(X) \in I$ and a morphism $f_X : X \to C_{i(X)}$ because $\mathscr{C}$ satisfies the solution set condition. If $i \in I$ then let $p_i : C_i \to W$ be the morphism associated to the coproduct W.

Let $q : W \to C$ be the morphism associated to the colimit $C = \text{colim}_{\mathbf{W}} D$. By composing, we obtain a morphism

$$X \xrightarrow{f_X} C_{i(X)} \xrightarrow{p_{i(X)}} W \xrightarrow{q} C.$$

Now, we will show that this is the unique element in $Hom_{\mathscr{C}}(X, C)$. To this end, assume that $f, g \in Hom_{\mathscr{C}}(X, C)$. Let $h = coeq(f, g) : C \to P$ be the coequalizer of f and g. By the same construction as before, we have a morphism $f_P : P \to C_{i(P)}$ and subsequently the composite

$$W \xrightarrow{q} C \xrightarrow{h} P \xrightarrow{f_P} C_{i(P)} \xrightarrow{p_{i(P)}} W.$$

Now apply the universal property of the colimit C to the above composite and id_W. We find that

$$q = q \circ (p_{i(P)} \circ f_P \circ h \circ q).$$

Since q is a "coequalizer" then it is a epimorphism in $\mathscr{C}$ (one can check this directly) and consequently

$$id_C = (q \circ p_{i(P)} \circ f_P) \circ h.$$

Composing both sides with h and then using the fact that $h = coeq(f, g)$ is an epimorphism in $\mathscr{C}$, we deduce that $id_P = h \circ (q \circ p_{i(P)} \circ f_p)$. So $h = coeq(f, g)$ is an isomorphism and thus, $f = g$. We conclude that if X is an object in $\mathscr{C}$ then $Hom_{\mathscr{C}}(X, C)$ has a unique element and C is a terminal object in $\mathscr{C}$ as required. □

The condition in Theorem 5.3.4 should remind us of the terminal object charactersation of an adjoint pair of functors in Theorem 2.3.2. As a result, the third and most general form of the adjoint functor theorem we will state and prove should not be too surprising.

Theorem 5.3.5 (Adjoint functor theorem V3)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Assume that $\mathscr{C}$ is also cocomplete (has small colimits). Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F has a right adjoint if and only if the following statement is satisfied: F preserves colimits and if X is an object in $\mathscr{D}$ then the comma category $(F, \mathbb{1}_X)$ (see equation (2.6)) satisfies the solution set condition.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are locally small categories, with $\mathscr{C}$ being cocomplete. Assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor. Firstly, assume that F has a right adjoint and is hence a left adjoint functor. Assume that X is an object in $\mathscr{D}$. By Theorem 5.1.1, F preserves colimits and by Theorem 2.3.2, the comma category $(F, \mathbb{1}_X)$ has a terminal object. By definition of the solution set condition, $(F, \mathbb{1}_X)$ satisfies the solution set condition (see part (a) of the proof of Theorem 5.3.4).

Conversely, assume that F preserves colimits and that if X is an object in $\mathscr{D}$ then the comma category $(F, \mathbb{1}_X)$ satisfies the solution set condition. By Theorem 5.3.4 and Theorem 2.3.2, it suffices to show that if X is an object in $\mathscr{D}$ then $(F, \mathbb{1}_X)$ has colimits. By Theorem 4.5.2, it suffices to show that $(F, \mathbb{1}_X)$ has coequalizers and arbitrary coproducts.

By Definition 2.3.2, the comma category $(F, \mathbb{1}_X)$ is the category whose objects are pairs (A, f) where A is an object in $\mathscr{C}$ and $f \in Hom_{\mathscr{D}}(F(A), X)$. A morphism $\alpha : (A, f) \to (A', g)$ is simply a morphism $\alpha : A \to A'$ in $\mathscr{C}$ which satisfies $g \circ F(\alpha) = f$. Hence, we have the projection functor

$$\begin{array}{cccc} P: & (F, \mathbb{1}_X) & \to & \mathscr{C} \\ & (A, f) & \mapsto & A \\ & \alpha : (A, f) \to (A', g) & \mapsto & \alpha : A \to A' \end{array}$$

In order to show that $(F, \mathbb{1}_X)$ has coequalizers and arbitrary coproducts, it suffices to show that P creates colimits by the dual result to Theorem 5.1.2. By Theorem 4.5.2, it suffices to show that P creates coequalizers and arbitrary coproducts.

First, we will show that P creates arbitrary coproducts. Assume that $\mathbf{I}$ is a small category with no non-identity morphisms and that we have a family $\{(A_i, f_i)\}_{i \in \mathbf{I}}$ of objects in $(F, \mathbb{1}_X)$. Let $c = \sqcup_{i \in \mathbf{I}} A_i$ be the coproduct of $\{A_i\}_{i \in \mathbf{I}}$ in $\mathscr{C}$. Since F preserves colimits, $F(c) = \sqcup_{i \in \mathbf{I}} F(A_i)$ is a coproduct in $\mathscr{D}$. By invoking its universal property, there exists a unique morphism $g : F(c) \to X$ such that if $i \in \mathbf{I}$ then the following diagram commutes:

$$\begin{array}{ccc} F(A_i) & \xrightarrow{\iota_{F(A_i)}} & F(c) \\ & \searrow^{f_i} & \downarrow g \\ & & X \end{array}$$

From this, it is straightforward to show that $(c, g) \in (F, \mathbb{1}_X)$ is the coproduct of $\{(A_i, f_i)\}_{i \in \mathbf{I}}$ by using the fact that c itself is a coproduct in $\mathscr{C}$. Noting that $P((c, g)) = c$, we deduce that P creates arbitrary coproducts. By a similar argument, P also creates coequalizers, completing the proof. □

Example 5.3.3. Here is an application of the dual result to Theorem 5.3.5 for limits. Let $F : \mathbf{Grp} \to \mathbf{Set}$. We will show that its associated left adjoint exists (the free group functor). First observe that **Grp** is a cocomplete category and that both **Grp** and **Set** are locally small categories. It is also easy to check that the forgetful functor F preserves limits.

By the dual result to Theorem 5.3.5, it suffices to show that if X is a set then the comma category $(\mathbb{1}_X, F)$ satisfies the solution set condition. The objects in this category are pairs (G, ϕ) where G is a group and $\phi : X \to F(G)$ is a morphism of sets. So let (G, ϕ) be an object in $(\mathbb{1}_X, F)$. The morphism of sets $\phi : X \to F(G)$ factors through the subgroup of G generated by the set $\{\phi(x) \mid x \in X\}$. Note that the cardinality of the subgroup must be bounded in terms of the cardinality of X.

Following this observation, we let I be the set of representatives of each isomorphism class of these subgroups. If $(G, \phi) \in (\mathbb{1}_X, F)$ then we take the representative R_ϕ associated to the isomorphism class of the subgroup $\langle\{\phi(x) \mid x \in X\}\rangle$ of G. If ι_ϕ is a group isomorphism from $\langle\{\phi(x) \mid x \in X\}\rangle$ to R_ϕ and $r_\phi : X \to F(R_\phi)$ satisfies $F(\iota_\phi) \circ \phi = r_\phi$ then

$$Hom_{(\mathbb{1}_X, F)}((G, \phi), (R_\phi, r_\phi)) \neq \emptyset$$

Therefore $(\mathbb{1}_X, F)$ satisfies the solution set condition and the left adjoint to F must exist.

Chapter 6

Monads

6.1 Motivation and definitions

There is a large amount of data to keep track of in a category (objects, morphisms, composition) and in some cases, it is difficult to understand, especially in the case where the category in question is not constructed in an explicit manner; for instance by a universal property. In this chapter, we will study the Barr-Beck theorem, which realises certain categories as categories of modules. The effect of this is that in order to understand the entire category, one only needs to understand the simpler category of modules and the algebra which acts on it. The algebra acting on it is called a *monad*, the central object of study in this chapter.

Let us begin by defining a monad.

Definition 6.1.1. Let $\mathscr{C}$ be a category. A **monad** acting on $\mathscr{C}$ is a triple (T, μ, η) consisting of the following data:

1. A functor $T : \mathscr{C} \to \mathscr{C}$.
2. A natural transformation $\mu : T \circ T \Rightarrow T$ called **multiplication**,
3. A natural transformation $\eta : id_{\mathscr{C}} \Rightarrow T$ called a **unit**.

Furthermore, the multiplication and unit must satisfy the following conditions:

1. **Associativity:** The following diagram commutes:

$$\begin{array}{ccc} T \circ T \circ T & \xrightarrow{id \circ \mu} & T \circ T \\ {\scriptstyle \mu \circ id}\downarrow & & \downarrow{\scriptstyle \mu} \\ T \circ T & \xrightarrow[\mu]{} & T \end{array}$$

2. **Unit:** The following diagrams commute:

$$\begin{array}{ccc} T \circ id_{\mathscr{C}} & \xrightarrow{id \circ \eta} & T \circ T \\ & \searrow^{=} & \downarrow{\scriptstyle \mu} \\ & & T \end{array}$$

$$\begin{array}{ccc} id_{\mathscr{C}} \circ T & \xrightarrow{\eta \circ id} & T \circ T \\ & \searrow^{=} & \downarrow{\scriptstyle \mu} \\ & & T \end{array}$$

In Definition 6.1.1, one is essentially imposing the conditions of a monoid on an element $T \in \mathcal{F}(\mathscr{C}, \mathscr{C})$. We remark here that Definition 6.1.1 is very similar to that of an *internal monoid*. In the definition of an internal monoid, one takes an element A of a finitely complete category $\mathscr{D}$ and imposes the conditions of a monoid on it. As one would expect, multiplication is a morphism $m : A \times A \to A$ and the unit is a morphism $e : * \to M$ where $*$ is the terminal object in $\mathscr{D}$. See [Bou17, Section 2.1] for more information on internal monoids.

Example 6.1.1. Let A be a unital associative algebra. The tensor functor

$$\begin{array}{rccc} T : & \mathbb{C}\textbf{-Vect} & \to & \mathbb{C}\textbf{-Vect} \\ & V & \mapsto & A \otimes V \end{array}$$

is a monad. The multiplication and the unit in this case arise from the multiplication map and the multiplicative unit in the algebra A respectively. Since A was conveniently assumed to be unital and associative then the multiplication and the unit must satisfy the properties outlined in the definition of a monad.

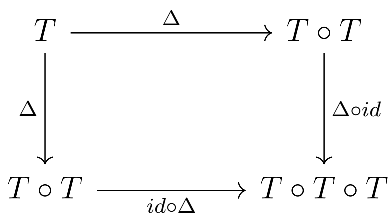
As usual, we can dualise the notion of a monad.

Definition 6.1.2. Let $\mathscr{C}$ be a category. A **comonad** acting on $\mathscr{C}$ is a triple (T, Δ, ϵ) consisting of the following data:

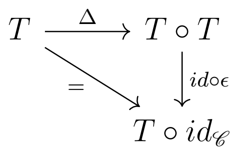
1. A functor $T : \mathscr{C} \to \mathscr{C}$.
2. A natural transformation $\Delta : T \Rightarrow T \circ T$ called **comultiplication**,
3. A natural transformation $\epsilon : T \Rightarrow id_{\mathscr{C}}$ called a **counit**.

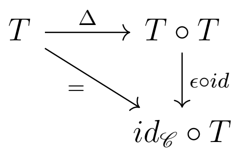
Furthermore, the comultiplication and counit must satisfy the following conditions:

1. **Coassociativity:** The following diagram commutes:



2. **Counit:** The following diagrams commute:





By definition, it is easy to see that comonads on a category $\mathscr{C}$ are monads on the opposite category $\mathscr{C}^{op}$. The next result provides us with a plentiful supply of monads and comonads and links the theory to adjoint pairs of functors.

Theorem 6.1.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. Suppose that the pair (F, G) is an adjoint pair of functors. Then $F \circ G$ has the structure of a comonad on $\mathscr{D}$ and $G \circ F$ has the structure of a monad on $\mathscr{C}$.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories. Assume that $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors such that (F, G) is an adjoint pair of functors. We will show that $G \circ F : \mathscr{C} \to \mathscr{C}$ has the structure of a monad on $\mathscr{C}$.

Since (F, G) is an adjoint pair then by Theorem 2.2.2, we have the unit of adjunction $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$ and the counit of adjunction $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$. Using the counit of adjunction, we obtain a natural transformation

$$\mu = id_G \circ \epsilon \circ id_F : G \circ (F \circ G) \circ F \Rightarrow G \circ F.$$

We claim that the triple $(G \circ F, \eta, \mu)$ is a monad on $\mathscr{C}$. To see that associativity holds, first observe that we have the following commutative diagram:

$$\begin{array}{ccc} FGFG & \xrightarrow{id_{F\circ G}\circ\epsilon} & FG \\ {\scriptstyle \epsilon\circ id_{F\circ G}}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ FG & \xrightarrow[\epsilon]{} & id_{\mathscr{D}} \end{array} \tag{6.1}$$

To see that the above diagram is commutative, it suffices to check it on an arbitrary object $X \in \mathscr{D}$. The resulting diagram in the category $\mathscr{D}$ is commutative because ϵ is a natural transformation.

Now take the diagram in equation (6.1), precompose with F and compose with G. We obtain the following commutative diagram:

$$\begin{array}{ccc} GFGFGF & \xrightarrow{id_{GFG}\circ\epsilon\circ id_F} & GFGF \\ {\scriptstyle \epsilon\circ id_{F\circ G}}\downarrow & & \downarrow{\scriptstyle id_{FG}\circ\epsilon} \\ GFGF & \xrightarrow[id_G\circ\epsilon\circ id_F]{} & GF \end{array}$$

Hence associativity is satisfied. To see that the unit properties are satisfied, we compute directly that if Y is an object in $\mathscr{C}$ then

$$\begin{aligned} (\mu \circ (id_{G\circ F} \circ \eta))((G \circ F)(Y)) &= (id_G \circ \epsilon \circ id_F) \circ (id_{G\circ F} \circ \eta)((G \circ F)(Y)) \\ &= (id_G \circ \epsilon \circ id_F) \circ \eta_{GF(Y)} \\ &= G(\epsilon_Y) \circ \eta_{GF(Y)} = id_{GF(Y)}. \end{aligned}$$

The last line utilises one of the two properties satisfied by η and ϵ in Theorem 2.2.2. Similarly, if Y is an object in $\mathscr{C}$ then

$$\begin{aligned} (\mu \circ (\eta \circ id_{G\circ F}))((G \circ F)(Y)) &= (id_G \circ \epsilon \circ id_F) \circ (\eta \circ id_{G\circ F})((G \circ F)(Y)) \\ &= (id_G \circ \epsilon \circ id_F) \circ \eta_{GF(Y)} \\ &= G(\epsilon_Y) \circ \eta_{GF(Y)} = id_{GF(Y)}. \end{aligned}$$

Hence, the unit properties in Definition 6.1.1 are satisfied and the triple $(G \circ F, \eta, \mu)$ is a monad on $\mathscr{C}$. Defining a comonad structure on $F \circ G$ proceeds via a very similar argument. □

6.2 Algebras over a monad

Just like how one can study a module over a monoid, one can define and study the analogous notion of an algebra over a monad.

Definition 6.2.1. Let $\mathscr{C}$ be a category and (T, η, μ) be a monad over $\mathscr{C}$. An **algebra** over T is a pair (X, α^X) consisting of the following data:

1. An object X in $\mathscr{C}$,
2. A morphism $\alpha^X : T(X) \to X$ in $\mathscr{C}$.

Moreover, the pair (X, α^X) must satisfy the following two properties:

1. **Associativity:** The following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} T^2(X) & \xrightarrow{T(\alpha^X)} & T(X) \\ {\scriptstyle \mu_X}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^X} \\ T(X) & \xrightarrow[\alpha^X]{} & X \end{array}$$

2. **Unit:** We have the equality

$$id_X = \alpha^X \circ \eta_X.$$

Definition 6.2.2. Let $\mathscr{C}$ be a category and (T, η, μ) be a monad over $\mathscr{C}$. The **category of T-algebras** in $\mathscr{C}$, denoted by $Alg_T(\mathscr{C})$, is the category whose objects are algebras over T. If (X, α^X) and (Y, α^Y) are objects in $Alg_T(\mathscr{C})$ then a morphism $f : (X, \alpha^X) \to (Y, \alpha^Y)$ is simply a morphism $f : X \to Y$ in $\mathscr{C}$ which makes the following diagram in $\mathscr{C}$ commute:

$$\begin{array}{ccc} T(X) & \xrightarrow{\alpha^X} & X \\ {\scriptstyle T(f)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T(Y) & \xrightarrow[\alpha^Y]{} & Y \end{array}$$

Dually, we also have the notion of a coalgebra over a comonad.

Definition 6.2.3. Let $\mathscr{C}$ be a category and (T, Δ, ϵ) be a comonad over $\mathscr{C}$. A **coalgebra** over T is a pair (X, β^X) consisting of the following data:

1. An object X in $\mathscr{C}$,
2. A morphism $\beta^X : X \to T(X)$ in $\mathscr{C}$.

Moreover, the pair (X, α^X) must satisfy the following two properties:

1. **Coassociativity:** The following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} X & \xrightarrow{\beta^X} & T(X) \\ {\scriptstyle \beta^X}\downarrow & & \downarrow{\scriptstyle \Delta_X} \\ T(X) & \xrightarrow[T(\beta^X)]{} & T^2(X) \end{array}$$

2. **Counit:** We have the equality

$$id_X = \epsilon_X \circ \beta^X.$$

Definition 6.2.4. Let $\mathscr{C}$ be a category and (T, Δ, ϵ) be a comonad over $\mathscr{C}$. The **category of T-coalgebras** in $\mathscr{C}$, denoted by $CoAlg_T(\mathscr{C})$, is the category whose objects are coalgebras over T. If (X, β^X) and (Y, β^Y) are objects in $CoAlg_T(\mathscr{C})$ then a morphism $f : (X, \beta^X) \to (Y, \beta^Y)$ is simply a morphism $f : X \to Y$ in $\mathscr{C}$ which makes the following diagram in $\mathscr{C}$ commute:

$$\begin{array}{ccc} X & \xrightarrow{\beta^X} & T(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow[\beta^Y]{} & T(Y) \end{array}$$

Example 6.2.1. Recall from Example 6.1.1 that if A is a unital associative algebra then the tensor functor

$$\begin{array}{rccc} T : & \mathbb{C}\textbf{-Vect} & \to & \mathbb{C}\textbf{-Vect} \\ & V & \mapsto & A \otimes V \end{array}$$

is a monad over the category $\mathbb{C}$-**Vect**. The category $Alg_T(\mathbb{C}\textbf{-Vect})$ turns out to be equivalent to the category A-**Alg** of A-algebras.

Example 6.2.2. Let X be a set and X^* be the set of finite length words whose letters are in X. The functor

$$\begin{array}{rccc} W: & \mathbf{Set} & \to & \mathbf{Set} \\ & X & \mapsto & X^* \end{array}$$

turns out to be a monad. If X is a set then we have a map

$$\begin{array}{rccc} \mu_X: & (W \circ W)(X) & \to & W(X) \\ & w_1 w_2 \dots w_n & \mapsto & w_1 \dots w_n \end{array}$$

given simply by concatenation of words. A word whose letters are themselves words from X is still a word in X. The family $(\mu_X)_{X \in \mathbf{Set}}$ endows W with multiplication $\mu : W \circ W \Rightarrow W$.

We also have a map $\eta_X : X \to W(X)$ which is the inclusion $X \hookrightarrow X^*$, thinking of X as one letter words. The family of maps $(\eta_X)_{X \in \mathbf{Set}}$ endows W with the unit $\eta : id_{\mathbf{Set}} \Rightarrow W$. It is straightforward to check that the triple (W, μ, η) is a monad over **Set**.

Now what is an algebra over the monad W? By definition, it is a pair (X, α^X) where X is a set and $\alpha^X : W(X) \to X$ is a morphism of sets satisfying associativity and the unit properties. To be clear, this means that

$$\alpha^X \circ \mu_X = \alpha^X \circ W(\alpha^X) \qquad \text{and} \qquad id_X = \alpha^X \circ \eta_X.$$

From this, the algebra (X, α^X) is simply the free monoid on X (see Definition 1.2.5).

The next theorem reveals that a (co)monad is *always* part of an adjunction.

Theorem 6.2.1. *Let $\mathscr{C}$ be a category and (T, μ, η) be a monad over $\mathscr{C}$. Then the forgetful functor $F : Alg_T(\mathscr{C}) \to \mathscr{C}$ has a left adjoint $G : \mathscr{C} \to Alg_T(\mathscr{C})$ and the monad $F \circ G$ from Theorem 6.1.1 is naturally isomorphic to T.*

Dually, let (U, Δ, ϵ) be a comonad over $\mathscr{C}$. Then the forgetful functor $F' : CoAlg_U(\mathscr{C}) \to \mathscr{C}$ has a right adjoint G' and the comonad $F' \circ G'$ from Theorem 6.1.1 is naturally isomorphic to U

Proof. Assume that $\mathscr{C}$ is a category and (T, μ, η) is a monad over $\mathscr{C}$. Define the functor

$$\begin{array}{rccc} G: & \mathscr{C} & \to & Alg_T(\mathscr{C}) \\ & X & \mapsto & (T(X), \mu_X). \end{array}$$

The pair $(T(X), \mu_X)$ is an algebra over T by definition. If X is an object in $\mathscr{C}$ then $(F \circ G)(X) = T(X)$. Thus it remains to show that the functor G is the desired left adjoint to the forgetful functor $F : Alg_T(\mathscr{C}) \to \mathscr{C}$.

To this end, assume that X is an object in $\mathscr{C}$ and (A, α^A) is a T-algebra. We will show that there is a natural isomorphism

$$Hom_{Alg_T(\mathscr{C})}(G(X), A) \cong Hom_{\mathscr{C}}(X, F(A)).$$

Let $\phi : (T(X), \mu_X) \to (A, \alpha^A)$ be a morphism of T-algebras. This is simply a morphism $\phi : T(X) \to A$ in $\mathscr{C}$ which makes the following diagram in $\mathscr{C}$ commute:

$$\begin{array}{ccc} T^2(X) & \xrightarrow{\mu_X} & T(X) \\ {\scriptstyle T(\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\ T(A) & \xrightarrow[\alpha^A]{} & A \end{array}$$

By using the unit property in Definition 6.1.1, we find that

$$\phi = \alpha^A \circ T(f) \circ T(\eta_X).$$

In particular, ϕ is uniquely determined by the composite

$$X \xrightarrow{\eta_X} T(X) \xrightarrow{f} A$$

which is a morphism in $\mathscr{C}$ from X to $F(A, \alpha^A) = A$. Consequently

$$Hom_{Alg_T(\mathscr{C})}(G(X), A) \cong Hom_{\mathscr{C}}(X, F(A))$$

and the pair (G, F) is an adjoint pair of functors. The dual argument proves the second part of the theorem. □

6.3 (Co)monadic functors and the Barr-Beck theorem

Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that (F, G) is an adjoint pair of functors. By Theorem 6.1.1, $T = G \circ F$ is a monad over $\mathscr{C}$ and $S = F \circ G$ is a comonad over $\mathscr{D}$.

Let $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$ be the unit and counit of adjunction respectively. Define the functors

$$\begin{array}{rccc} G_{enh}: & \mathscr{D} & \to & Alg_T(\mathscr{C}) \\ & Y & \mapsto & (G(Y), G(\epsilon_Y)) \end{array} \tag{6.2}$$

and

$$\begin{array}{rccc} F_{enh}: & \mathscr{C} & \to & CoAlg_S(\mathscr{D}) \\ & Z & \mapsto & (F(Z), F(\eta_Z)). \end{array} \tag{6.3}$$

A few lengthly but simple computations are required to show that the functors in equation (6.2) and equation (6.3) are well-defined functors.

Definition 6.3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $G : \mathscr{D} \to \mathscr{C}$ be a functor. We say that G is **monadic** if the following two statements are satisfied:

1. G has a left adjoint $F : \mathscr{C} \to \mathscr{D}$.
2. If $T = G \circ F$ is the associated monad to the adjoint pair (F, G) (see Theorem 6.1.1) and $G_{enh} : \mathscr{D} \to Alg_T(\mathscr{C})$ is the functor in equation (6.2) then G_{enh} is an equivalence.

Definition 6.3.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F is **comonadic** if the following two statements are satisfied:

1. F has a right adjoint $G : \mathscr{D} \to \mathscr{C}$.
2. If $S = F \circ G$ is the associated comonad to the adjoint pair (F, G) (see Theorem 6.1.1) and $F_{enh} : \mathscr{C} \to CoAlg_S(\mathscr{D})$ is the functor in equation (6.3) then F_{enh} is an equivalence.

The main result of this section is the *Barr-Beck theorem* which gives necessary and sufficient conditions for a functor to be monadic. In order to understand the statement of the Barr-Beck theorem, we will now deal with the necessary prerequisites.

Lemma 6.3.1. *Let $\mathscr{C}$ be a category and $\boldsymbol{I}$ be a small category. Let (T, μ, η) be a monad over $\mathscr{C}$ and $F : Alg_T(\mathscr{C}) \to \mathscr{C}$ be the forgetful functor. Then*

1. *F creates limits,*
2. *If T preserves colimits of shape $\boldsymbol{I}$ then F reflects colimits of shape $\boldsymbol{I}$.*

Proof. Assume that (T, μ, η) is a monad over $\mathscr{C}$ and that $F : Alg_T(\mathscr{C}) \to \mathscr{C}$ is the forgetful functor.

To show: (a) F creates limits.

(a) Assume that $J : \mathbf{I} \to Alg_T(\mathscr{C})$ is a diagram and that the family of morphisms

$$\big(p_I : L \to (F \circ J)(I)\big)_{I \in \mathbf{I}}$$

is a limit of $F \circ J$. We have to show that there is a limit of J which maps to the above limit via F. Assume that I is an object in $\mathbf{I}$. Then $J(I)$ is a T-algebra with associated morphism $\alpha^{J(I)} : T(J(I)) \to J(I)$. Now consider the composite

$$q_I = \alpha^{J(I)} \circ T(p_I) : T(L) \to (F \circ J)(I).$$

The family of morphisms $(q_I)_{I \in \mathbf{I}}$ defines a cone on $F \circ J$. By the universal property of the limit, there exists a unique morphism $\alpha^L : T(L) \to L$ such that the following diagram in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} T(L) & \overset{\alpha^L}{\dashrightarrow} & L \\ {\scriptstyle T(p_I)}\downarrow & & \downarrow{\scriptstyle p_I} \\ T((F \circ J)(I)) & \underset{\alpha^{J(I)}}{\longrightarrow} & (F \circ J)(I) \end{array} \tag{6.4}$$

Now we will show that the pair (L, α^L) is a T-algebra. To see that (L, α^L) satisfies the associativity property in Definition 6.2.1, we observe that if I is an object in $\mathbf{I}$ then

$$\begin{aligned} p_I \circ (\alpha^L \circ T(\alpha^L)) &= \alpha^{J(I)} \circ T(p_I) \circ T(\alpha^L) \qquad \text{(by diagram (6.4))} \\ &= \alpha^{J(I)} \circ T(p_I \circ \alpha^L) \\ &= \alpha^{J(I)} \circ T(\alpha^{J(I)} \circ T(p_I)) \qquad \text{(by diagram (6.4))} \\ &= (\alpha^{J(I)} \circ T(\alpha^{J(I)})) \circ T^2(p_I) \\ &= \alpha^{J(I)} \circ \mu_{J(I)} \circ T^2(p_I) \qquad \text{(by Definition 6.2.1)} \\ &= \alpha^{J(I)} \circ T(p_I) \circ \mu_L \qquad \text{(naturality of } \mu) \\ &= p_I \circ (\alpha^L \circ \mu_L). \end{aligned}$$

By Lemma 4.6.3, $\alpha^L \circ T(\alpha^L) = \alpha^L \circ \mu_L$ and associativity is satisfied. To see that the unit property in Definition 6.2.1 is satisfied,

$$
\begin{aligned}
p_I \circ (\alpha^L \circ \eta_L) &= \alpha^{J(I)} \circ (T(p_I) \circ \eta_L) \qquad \text{(by diagram (6.4))} \\
&= \alpha^{J(I)} \circ (\eta_{J(I)} \circ p_I) \qquad \text{(by naturality of } \eta) \\
&= p_I = p_I \circ id_L. \qquad \text{(by Definition 6.2.1)}
\end{aligned}
$$

By Lemma 4.6.3, (L, α^L) satisfies the unit property and hence, is a T-algebra. To see that (L, α^L) is a limit of J, first observe that by commutativity of diagram (6.4), we have a family of morphisms

$$\left(p_I : (L, \alpha^L) \to (J(I), \alpha^{J(I)}) = J(I)\right)_{I \in \mathbf{I}}$$

Now suppose that we have another cone over J:

$$\left(q_I : (X, \alpha^X) \to J(I)\right)_{I \in \mathbf{I}}$$

By the universal property of the limit $(p_I)_{I \in \mathbf{I}}$ in $\mathscr{C}$, there exists a unique morphism $\psi : X \to L$ in $\mathscr{C}$ such that if I is an object in $\mathbf{I}$ then $p_I \circ \psi = q_I$. So

$$
\begin{aligned}
p_I \circ \psi \circ \alpha^X &= q_I \circ \alpha^X \\
&= \alpha^{J(I)} \circ T(q_I) \qquad \text{(by Definition 6.2.1)} \\
&= \alpha^{J(I)} \circ T(p_I) \circ T(\psi) \\
&= p_I \circ \alpha^L \circ T(\psi). \qquad \text{(by diagram (6.4))}
\end{aligned}
$$

By Lemma 4.6.3, $\psi \circ \alpha^X = \alpha^L \circ T(\psi)$ and ψ is a morphism in $Alg_T(\mathscr{C})$ from (X, α^X) to (L, α^L). Therefore $(p_I)_{I \in \mathbf{I}}$ is a limit of J which is obviously sent to the limit $(p_I)_{I \in \mathbf{I}}$ of $F \circ J$ via F. So F creates limits thereby proving part (a).

Now assume that T preserves colimits of shape $\mathbf{I}$. Let $C = \operatorname{colim}_{\mathbf{I}}(F \circ J)$ be the colimit associated to $F \circ J$. The composite

$$T(\operatorname{colim}_{\mathbf{I}}(F \circ J)) \xrightarrow{\cong} \operatorname{colim}_{\mathbf{I}}\big(T \circ (F \circ J)\big) \xrightarrow{\operatorname{colim}_{\mathbf{I}}(\alpha^{J(-)})} \operatorname{colim}_{\mathbf{I}}(F \circ J)$$

supplies the necessary structure to turn C into a T-algebra. This requires computations we will omit here. By the universal property of the colimit applied to C, we deduce that C is also a colimit of J in $Alg_T(\mathscr{C})$. Hence F reflects colimits of shape $\mathbf{I}$. This completes the proof. □

Next, we define the notion of a conservative functor.

Definition 6.3.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F is **conservative** if the following statement is satisfied: If f is a morphism in $\mathscr{C}$ and $F(f)$ is an isomorphism in $\mathscr{D}$ then f is an isomorphism in $\mathscr{C}$.

Example 6.3.1. Let k be a field. Then the forgetful functors k-**Vect** $\to$ **Set** and **Grp** $\to$ **Set** are conservative functors.

Example 6.3.2. It is well-known that the category of groups **Grp** satisfies the short five lemma (see [Wei94, Exercise 1.3.3]). The notion of a *protomodular category* generalises this. A finitely complete category with a zero object is said to be protomodular if it satisfies a weaker version of the short five lemma, called the split short five lemma (see [BB04, Definition 3.1.1]).

In [BB04, Proposition 3.1.2], an equivalent characterisation for a finitely complete category with zero object $\mathscr{C}$ to be protomodular is proved, which states that the *inverse image functor* (or base change functor in [Bou17]) associated to an arbitrary morphism in $\mathscr{C}$ is conservative. Here are a few examples of protomodular categories: **Grp**, **Ab**, abelian categories and **Grp(Top)**; the category of topological groups. By the characterisation in [BB04, Proposition 3.1.2], these examples of protomodular categories give rise to a plentiful supply of conservative functors. For more information on protomodular categories, consult [Bou17] and [BB04, Chapter 3].

The following lemma examines what happens when conservative functors have left/right adjoints.

Lemma 6.3.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a conservative functor. If F has a fully faithful left adjoint or a fully faithful right adjoint then F is an equivalence of categories.*

Proof. Assume that $F : \mathscr{C} \to \mathscr{D}$ is a conservative functor which has a fully faithful left adjoint $G : \mathscr{D} \to \mathscr{C}$. Let

$$\eta : id_{\mathscr{D}} \Rightarrow F \circ G \qquad \text{and} \qquad \epsilon : G \circ F \Rightarrow id_{\mathscr{C}}$$

be the unit and counit of adjunction associated to the adjoint pair (G, F). Since G is fully faithful then η is a natural isomorphism. To see that F is an equivalence of categories, it suffices to show that the counit ϵ is also a natural isomorphism.

To this end, assume that X is an object in $\mathscr{C}$. Using Theorem 2.2.1, we observe that the composite

$$F(X) \xrightarrow{\eta_{F(X)}} FGF(X) \xrightarrow{F(\epsilon_X)} F(X)$$

is equal to the identity morphism $id_{F(X)}$. Since $\eta_{F(X)}$ is an isomorphism then $F(\epsilon_X)$ is an isomorphism in $\mathscr{D}$. Since F is a conservative functor then ϵ_X is an isomorphism in $\mathscr{C}$ and ϵ is a natural isomorphism. Consequently, F is an equivalence of categories as required. An analogous argument proves that F is an equivalence if it admits a fully faithful right adjoint. □

Next, we will define the notion of a fork, which is closely related to the notion of a coequalizer.

Definition 6.3.4. Let $\mathscr{C}$ be a category. A **fork** is a diagram in $\mathscr{C}$

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{q} Z \tag{6.5}$$

In other words, a fork is a cocone of a diagram $D : \mathbf{I} \to \mathscr{C}$ where $\mathbf{I}$ is the small category

$$\bullet \rightrightarrows \bullet$$

We say that the fork in equation (6.5) is **split** if there exist morphisms $s : Z \to Y$ and $t : Y \to X$ in $\mathscr{C}$ such that

$$q \circ s = id_Z, \qquad f \circ t = id_Y \qquad \text{and} \qquad g \circ t = s \circ q.$$

It is straightforward to check that a split fork is a coequalizer. This leads us to the next definition.

Definition 6.3.5. Let $\mathscr{C}$ be a category and $f, g : X \to Y$ be morphisms in $\mathscr{C}$. We say that the pair (f, g) is a **split pair** if the coequalizer $q = coeq(f, g)$ exists and the resulting fork

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{q} Z$$

is split.

Definition 6.3.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Let f, g be morphisms in $\mathscr{C}$. We say that the pair (f, g) is F**-split** if the pair of morphisms $(F(f), F(g))$ in $\mathscr{D}$ is a split pair.

The point of the above definitions is that monads give rise to examples of split pairs.

Example 6.3.3. Let $\mathscr{C}$ be a category and (T, μ, η) be a monad over $\mathscr{C}$. Let (X, α^X) be a T-algebra. We claim that the pair $(\mu_X, T(\alpha^X))$ is a split pair. By the associativity property in Definition 6.2.1, we have a fork

$$T^2(X) \underset{T(\alpha^X)}{\overset{\mu_X}{\rightrightarrows}} T(X) \xrightarrow{\alpha^X} X$$

To see that this fork is split, note that we have the morphisms

$$\eta_X : X \to T(X) \qquad \text{and} \qquad \eta_{T(X)} : T(X) \to T^2(X).$$

By Definition 6.2.1, we have $id_X = \alpha^X \circ \eta_X$. By Definition 6.1.1, $id_{T(X)} = \mu_X \circ \eta_{T(X)}$. Finally,

$$T(\alpha^X) \circ \eta_{T(X)} = \eta_X \circ \alpha^X$$

because $\eta : id_{\mathscr{C}} \Rightarrow T$ is a natural transformation.

Example 6.3.4. Now let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that (F, G) is an adjoint pair. Let Y be an object in $\mathscr{D}$ and $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$ be the counit of adjunction. Then we have a fork

$$FGFG(Y) \underset{FG(\epsilon_Y)}{\overset{\epsilon_{FG(Y)}}{\rightrightarrows}} FG(Y) \xrightarrow{\epsilon_Y} Y$$

If we apply the functor G to the above fork, we obtain another fork

$$GFGFG(Y) \underset{GFG(\epsilon_Y)}{\overset{G(\epsilon_{FG(Y)})}{\rightrightarrows}} GFG(Y) \xrightarrow{G(\epsilon_Y)} G(Y)$$

Now recall from Theorem 6.1.1 that the triple $(G \circ F, \eta, \zeta)$ is a monad over $\mathscr{C}$ where ζ is the natural transformation

$$\zeta = id_G \circ \epsilon \circ id_F : G \circ (F \circ G) \circ F \Rightarrow G \circ F.$$

Hence by the previous example (Example 6.3.3), the above fork is split. Therefore the pair $(\epsilon_{FG(Y)}, FG(\epsilon_Y))$ is G-split.

Now we have arrived at the main theorem of this chapter.

Theorem 6.3.3 (Barr-Beck)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $G : \mathscr{D} \to \mathscr{C}$ be a functor. Then G is a monadic functor if and only if the following statements are satisfied:*

1. *G has a left adjoint,*

2. *G is a conservative functor,*

3. *If (f, g) is a G-split pair of morphisms in $\mathscr{D}$ then it has a coequalizer $coeq(f, g)$ and $G(coeq(f, g)) = coeq(G(f), G(g))$.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are categories and that $G : \mathscr{D} \to \mathscr{C}$ is a functor. First assume that G is a monadic functor. By definition of a monadic functor, G has a left adjoint $F : \mathscr{C} \to \mathscr{D}$.

To show: (a) G is conservative.

(b) If (f, g) is a G-split pair of morphisms in $\mathscr{D}$ then the coequalizer $coeq(f, g)$ exists.

(c) $G(coeq(f, g)) = coeq(G(f), G(g))$.

(a) Let $T = G \circ F$ be the associated monad to the adjoint pair (F, G) (see Theorem 6.1.1). Let $G_{enh} : \mathscr{D} \to Alg_T(\mathscr{C})$ be the functor defined in equation (6.2). Since G is a monadic functor then G_{enh} is an equivalence.

Now let $\mathcal{F}_{\mathscr{C}} : Alg_T(\mathscr{C}) \to \mathscr{C}$ be the forgetful functor. By equation (6.2),

$$\mathcal{F}_{\mathscr{C}} \circ G_{enh} = G.$$

Now since G_{enh} is an equivalence, G is conservative if and only if $\mathcal{F}_{\mathscr{C}}$ is conservative. So, we will show that $\mathcal{F}_{\mathscr{C}}$ is a conservative functor. To this end, assume that (X, α^X) and (Y, α^Y) are objects in $Alg_T(\mathscr{C})$ and $f : (X, \alpha^X) \to (Y, \alpha^Y)$ is a morphism in $Alg_T(\mathscr{C})$. Then f is a morphism in $\mathscr{C}$ from X to Y satisfying

$$f \circ \alpha^X = \alpha^Y \circ T(f).$$

Suppose further that $f = \mathcal{F}_{\mathscr{C}}(f)$ is an isomorphism in $\mathscr{C}$. Then

$$f \circ (f^{-1} \circ \alpha^Y) = \alpha^Y = (\alpha^Y \circ T(f)) \circ T(f)^{-1} = f \circ (\alpha^X \circ T(f)^{-1}).$$

Since f is an isomorphism then f is a monomorphism and consequently, $f^{-1} \circ \alpha^Y = \alpha^X \circ T(f)^{-1}$. So f^{-1} defines a morphism of T-algebras from (Y, α^Y) to (X, α^X) which is inverse to $f \in Alg_T(\mathscr{C})$. Thus, f is an isomorphism in $Alg_T(\mathscr{C})$ and the forgetful functor $\mathcal{F}_{\mathscr{C}}$ is a conservative functor.

(b) Assume that (f, g) is a G-split pair of morphisms in $\mathscr{D}$ from X to Y. Then the pair $(G(f), G(g))$ in $\mathscr{C}$ is a split pair. Let $h : G(Y) \to A$ be the coequalizer of $(G(f), G(g))$. Then the fork in $\mathscr{C}$

$$G(X) \underset{G(g)}{\overset{G(f)}{\rightrightarrows}} G(Y) \xrightarrow{h} A$$

is split, which means that there exist morphisms $s : A \to G(Y)$ and $t : G(Y) \to G(X)$ in $\mathscr{C}$ such that

$$h \circ s = id_A, \qquad G(f) \circ t = id_{G(Y)} \qquad \text{and} \qquad G(g) \circ t = s \circ h.$$

The idea is to apply the functor G_{enh} to the pair (f, g). Let $\epsilon : F \circ G \Rightarrow id_{\mathscr{D}}$ be the counit of adjunction associated to the adjoint pair (F, G). By equation (6.2),

$$G_{enh}(X) = (G(X), G(\epsilon_X)) \qquad \text{and} \qquad G_{enh}(Y) = (G(Y), G(\epsilon_Y)).$$

We also have two morphisms of T-algebras $G_{enh}(f)$ and $G_{enh}(g)$. The claim here is that $A \in \mathscr{C}$ is actually a T-algebra and that $h : G(Y) \to A$ is a morphism of T-algebras.

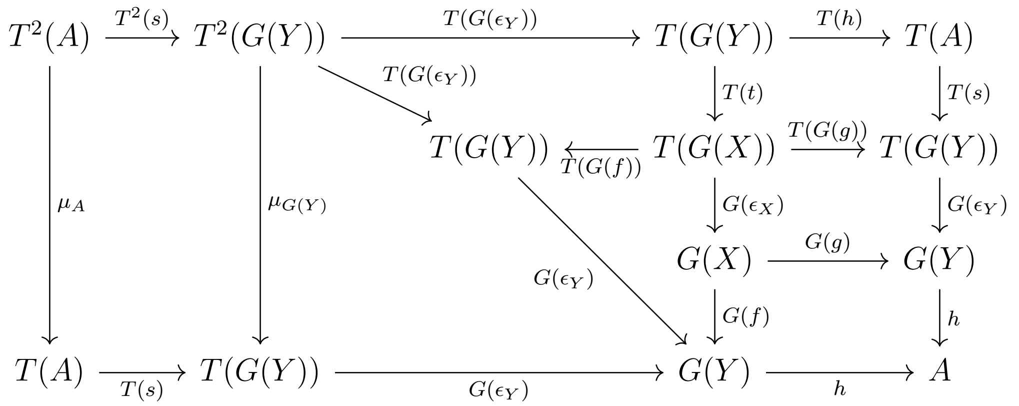
To show: (ba) A is a T-algebra.

(bb) $h : G(Y) \to A$ is a morphism of T-algebras.

(ba) Define the morphism $\alpha^A : T(A) \to A$ by the composite

$$T(A) \xrightarrow{T(s)} T(G(Y)) \xrightarrow{G(\epsilon_Y)} G(Y) \xrightarrow{h} A.$$

We claim that (A, α^A) is a T-algebra. To see that associativity is satisfied, consider the following diagram in $\mathscr{C}$:



(6.6)

We want to show that the outer square in diagram (6.6) is commutative. The point here is that by the properties of split forks, T-algebras and naturality of the multiplication map $\mu : T^2 \Rightarrow T$, every subdiagram of diagram (6.6) commutes. Thus, the outer square of diagram (6.6) commutes and α^A satisfies associativity. By another diagram chase, every subdiagram of the diagram below

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & T(A) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle T(s)} \\ G(Y) & \xrightarrow{\eta_{G(Y)}} & T(G(Y)) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle G(\epsilon_Y)} \\ A & \xleftarrow[h]{} & G(Y) \end{array}$$

commutes. Recall that $\eta : id_{\mathscr{C}} \Rightarrow T = G \circ F$ is not only the unit associated to the monad T over $\mathscr{C}$ — it is also the unit of adjunction associated to the adjoint pair (F, G) (see Theorem 6.1.1). Note that in the context of the above diagram, $G(\epsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)}$. The top square commutes by naturality of η. Commutativity of the outer square tells us that (A, α^A) satisfies the unit property and is therefore a T-algebra.

(bb) By examining the rectangle formed by the last two columns of diagram (6.6), we deduce that h is a morphism of T-algebras from $(G(Y), G(\epsilon_Y))$ to (A, α^A).

(b) We now claim that h is the coequalizer of the pair $(G_{enh}(f), G_{enh}(g))$ in $Alg_T(\mathscr{C})$. Assume that (W, α^W) is a T-algebra and that $e' : G(Y) \to W$ is a morphism in $\mathscr{C}$ satisfying $e' \circ G(f) = e' \circ G(g)$. Since h is the coequalizer of the pair $(G(f), G(g))$ then there exists a unique morphism $j : A \to W$ in $\mathscr{C}$ such that $e' = j \circ h$.

It remains to show that j is a morphism of T-algebras. Consider the following diagram in $\mathscr{C}$:

$$\begin{array}{ccccccc} T(A) & \xrightarrow{T(s)} & T(G(Y)) & \xrightarrow{G(\epsilon_Y)} & G(Y) & \xrightarrow{h} & A \\ & & \downarrow{\scriptstyle T(e')} & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle s} \\ & & T(W) & \xrightarrow[\alpha^W]{} & W & \xleftarrow[e']{} & G(Y) \end{array}$$

We want to show that $j \circ \alpha^A = \alpha^W \circ T(j)$. The morphism α^A is the top row of the above diagram. Observe that since $h \circ s = id_A$ then $j = e' \circ s$. Hence the relation $j \circ \alpha^A = \alpha^W \circ T(j)$ is equivalent to commutativity of the above diagram. The LHS square is commutative because e' is a morphism of T-algebras and the RHS square is commutative because

$$e' \circ s \circ h = j \circ h = e'.$$

Therefore h is the coequalizer of $(G_{enh}(f), G_{enh}(g))$ in $Alg_T(\mathscr{C})$. Since $G_{enh} : D \to Alg_T(\mathscr{C})$ is an equivalence by assumption, we obtain a coequalizer $coeq(f, g)$ of the pair (f, g) in $\mathscr{D}$. This completes the proof of part (b).

(c) Now observe that

$$coeq(G(f), G(g)) = h = \mathcal{F}_{\mathscr{C}}(h) = (\mathcal{F}_{\mathscr{C}} \circ G_{enh})(coeq(f, g)) = G(coeq(f, g)).$$

With parts (b) and (c) of the proof, we demonstrated that the third condition in the statement of the Barr-Beck theorem is satisfied.

Conversely, assume that G satisfies the three conditions in the statement of the Barr-Beck theorem. By assumption G has a left adjoint. So it suffices to prove that the functor $G_{enh} : \mathscr{D} \to Alg_T(\mathscr{C})$ is an equivalence. By Lemma 6.3.2, it suffices to show that G_{enh} is conservative and admits a fully faithful left adjoint.

To show: (d) G_{enh} is a conservative functor.

(e) G_{enh} has a fully faithful left adjoint.

(d) By assumption, G is a conservative functor. Assume that $f : X \to Y$ is a morphism in $\mathscr{D}$. Assume that

$$G_{enh}(f) : (G(X), G(\epsilon_X)) \to (G(Y), G(\epsilon_Y))$$

is an isomorphism in $Alg_T(\mathscr{C})$. Then there exists a morphism of T-algebras $g : (G(Y), G(\epsilon_Y)) \to (G(X), G(\epsilon_X))$ such that $f \circ g = id_{G_{enh}(Y)}$ and $g \circ f = id_{G_{enh}(X)}$. Notably, $g : G(Y) \to G(X)$ is a morphism in $\mathscr{D}$ such that $f \circ g = id_{G(Y)}$ and $g \circ f = id_{G(X)}$. Therefore, f is an isomorphism in $\mathscr{D}$ and G_{enh} is a conservative functor as required.

(e) By Theorem 6.2.1, the forgetful functor $\mathcal{F}_{\mathscr{C}} : Alg_T(\mathscr{C}) \to \mathscr{C}$ admits a left adjoint functor

$$\begin{array}{rccc} \mathcal{F}^L_{\mathscr{C}} : & \mathscr{C} & \to & Alg_T(\mathscr{C}) \\ & X & \mapsto & (T(X), \eta_X). \end{array}$$

If X is an object in $\mathscr{C}$ then we know that if a functor

$$H : Alg_T(\mathscr{C}) \to \mathscr{D}$$

is a left adjoint to G_{enh} then it must satisfy

$$H(T(X)) = (H \circ G_{enh})(F(X)) \cong F(X).$$

The idea behind the proof of part (e) is to "resolve" every T-algebra by free T-algebras (T-algebras of the form $\mathcal{F}^L_{\mathscr{C}}(X)$) and then use the above characterisation of H on free T-algebras to construct a genuine left adjoint functor to G_{enh} on all T-algebras.

Define the functor $G^L_{enh} : Alg_T(\mathscr{C}) \to \mathscr{D}$ to be the composite

$$Alg_T(\mathscr{C}) \xrightarrow{\mathcal{F}_{\mathscr{C}}} \mathscr{C} \xrightarrow{F} \mathscr{D}.$$

If X is an object in $\mathscr{C}$ then

$$G^L_{enh}((T(X), \eta_X)) = (F \circ \mathcal{F}_{\mathscr{C}} \circ \mathcal{F}^L_{\mathscr{C}})(T(X)) \cong F((G \circ F)(X)) \cong F(X).$$

Hence, our construction of a left adjoint to G_{enh} will begin with G^L_{enh}. Define $\mathcal{A} \subsetneq Alg_T(\mathscr{C})$ to be the full subcategory satisfying the following two properties:

1. If $X \in \mathcal{A}$ then the functor

$$\begin{array}{rccc} \mathcal{R}_X : & \mathscr{D} & \to & \textbf{Set} \\ & Y & \mapsto & Hom_{Alg_T(\mathscr{C})}(X, G_{enh}(Y)) \end{array}$$

   is corepresentable (see Definition 3.2.1),

2. If X is an object in $\mathcal{A}$ then the unit morphism $\zeta_X : X \to (G_{enh} \circ G^L_{enh})(X)$ induces an isomorphism

$$\mathcal{F}_{\mathscr{C}}(X) \to \mathcal{F}^L_{\mathscr{C}}((G_{enh} \circ G^L_{enh})(X)) \cong G(G^L_{enh}(X))$$

   in $\mathscr{C}$.

By our construction of $\mathcal{A}$, the restriction $G^L_{enh}|_{\mathcal{A}}$ gives a partial left adjoint to G_{enh}, yielding the natural bijection

$$Hom_{\mathscr{D}}(G^L_{enh}(X), Y) \cong Hom_{Alg_T(\mathscr{C})}(X, G_{enh}(Y))$$

where X is an object in $\mathcal{A}$ and Y is an object in $\mathscr{D}$. Observe that the essential image of $\mathcal{F}^L_{\mathscr{C}}$ is in $\mathcal{A}$. This is because

$$Hom_{Alg_T(\mathscr{C})}(\mathcal{F}^L_{\mathscr{C}}(X), G_{enh}(Y)) \cong Hom_{\mathscr{C}}(X, G(Y)) \cong Hom_{\mathscr{D}}(F(X), Y).$$

and if X is an object in $\mathscr{C}$ then the unit morphism

$$\zeta_{\mathcal{F}^L_{\mathscr{C}}(X)} : \mathcal{F}^L_{\mathscr{C}}(X) \to (G_{enh} \circ G^L_{enh})(\mathcal{F}^L_{\mathscr{C}}(X))$$

induces an isomorphism in $\mathscr{C}$ from $\mathcal{F}_{\mathscr{C}}(\mathcal{F}^L_{\mathscr{C}}(X)) \cong X$ to

$$(\mathcal{F}_{\mathscr{C}} \circ G_{enh} \circ G^L_{enh})(\mathcal{F}^L_{\mathscr{C}}(X)) \cong GF(X) = T(X).$$

Now recall from Example 6.3.3 that if (X, α^X) is a T-algebra then it fits into a split fork (and hence a coequalizer)

$$T^2(X) \underset{T(\alpha^X)}{\overset{\mu_X}{\rightrightarrows}} T(X) \xrightarrow{\alpha^X} X$$

of free T-algebras. Now let X_i be the diagram

$$T^2(X) \underset{T(\alpha^X)}{\overset{\mu_X}{\rightrightarrows}} T(X)$$

in $Alg_T(\mathscr{C})$ By applying the forgetful functor $\mathcal{F}_{\mathscr{C}}$ to X_i, we find that

$$\mathcal{F}_{\mathscr{C}}(X_i) \cong (G \circ G^L_{enh})(X_i)$$

is a split coequalizer in $\mathscr{C}$. The above isomorphism follows from the induced isomorphism

$$\mathcal{F}_{\mathscr{C}}(\mathcal{F}^L_{\mathscr{C}}(Z)) \cong (\mathcal{F}_{\mathscr{C}} \circ G_{enh} \circ G^L_{enh})(\mathcal{F}^L_{\mathscr{C}}(Z)) = (G \circ G^L_{enh})(\mathcal{F}^L_{\mathscr{C}}(Z)).$$

for $Z \in \mathscr{C}$. To see this, apply the isomorphism to X and $T(X)$. Consequently, the pair of morphisms $(G^L_{enh}(\mu_X), G^L_{enh}(T(\alpha^X)))$ in $\mathscr{D}$ is G-split. By third property, this pair must have a coequalizer in $\mathscr{D}$.

Now if (X, α^X) is an object in $Alg_T(\mathscr{C})$ and Y is an object in $\mathscr{D}$ then $X \cong \mathrm{colim}_i X_i$ and

$$\begin{aligned} Hom_{Alg_T(\mathscr{C})}(X, G_{enh}(Y)) &\cong Hom_{Alg_T(\mathscr{C})}(\mathrm{colim}_i X_i, G_{enh}(Y)) \\ &\cong \lim_i Hom_{Alg_T(\mathscr{C})}(X_i, G_{enh}(Y)) \\ &\cong \lim_i Hom_{Alg_T(\mathscr{C})}(G^L_{enh}(X_i), Y) \\ &\cong Hom_{Alg_T(\mathscr{C})}(\mathrm{colim}_i G^L_{enh}(X_i), Y). \end{aligned}$$

The third isomorphism follows from the fact that the diagram X_i is in $\mathcal{A}$ (it is comprised of free T-algebras). The second and fourth isomorphisms follow from Example 5.1.1 (the Yoneda functor maps colimits to limits).

Here is another obsevation. If (X, α^X) is an object in $Alg_T(\mathscr{C})$ then the induced isomorphism in $\mathscr{C}$

$$X = \mathcal{F}_{\mathscr{C}}(X) \to (G \circ G^L_{enh})(X)$$

is given by the composite

$$\mathrm{colim}_i X_i \cong \mathrm{colim}_i (G \circ G^L_{enh})(X_i) \to G(\mathrm{colim}_i G^L_{enh}(X_i)) \cong (G \circ G^L_{enh})(X).$$

This is an isomorphism because $G^L_{enh}(X_i)$ (or more explicitly $(G^L_{enh}(\mu_X), G^L_{enh}(T(\alpha^X))))$ is G-split and G preserves such coequalizers.

Therefore, we have a left adjoint functor $G^L_{enh} : Alg_T(\mathscr{C}) \to \mathscr{D}$ to G_{enh} such that the unit morphism $\zeta_X : X \to (G_{enh} \circ G^L_{enh})(X)$ is an isomorphism in $\mathscr{C}$ after applying the forgetful functor $\mathcal{F}_{\mathscr{C}}$. Now $\mathcal{F}_{\mathscr{C}}$ is conservative by the proof of part (a). So, G^L_{enh} is also fully faithful.

By combining parts (d) and (e), we deduce that G_{enh} is an equivalence and hence G is a monadic functor as required. $\square$

Here is the dual statement to the Barr-Beck theorem in Theorem 6.3.3.

Theorem 6.3.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F is a comonadic functor if and only if the following statements are satisfied:*

1. *F has a right adjoint,*
2. *F is a conservative functor,*
3. *If (f, g) is a F-cosplit pair of morphisms in $\mathscr{D}$ then it has a coequalizer $eq(f, g)$ and $F(eq(f, g)) = eq(F(f), F(g))$.*

6.4 The Barr-Beck theorem in descent theory

As in [Saf23, Section 5.4], we will provide an idea of how Theorem 6.3.4 can be applied to the problem of descent in Galois theory (the study of field extensions) and more generally, to extensions of rings in commutative algebra. We remark that in the statement of Theorem 6.3.4, the notion of a F-cosplit pair has not been defined formally. It is the dual notion to a split pair, as defined in the previous section. Fortunately in this section, all the categories discussed will have equalizers and the all the functors will preserve them. Hence, we need not worry about the F-cosplit condition.

To illustrate the notion of descent, let us begin with the simplest case of descent for functions. Let X be a set and k be a field. Define $\mathcal{O}(X)$ to be the commutative k-algebra of functions $f : X \to k$. Then we have a contravariant functor

$$\begin{array}{cccc} \mathcal{O} : & \mathbf{Set}^{op} & \to & \mathbf{Com}\ k\text{-}\mathbf{Alg} \\ & X & \mapsto & \mathcal{O}(X) \\ & \phi : X \to Y & \mapsto & \big(f \mapsto f \circ \phi\big) \end{array}$$

Since $\mathcal{O}$ is an algebra of functions then $\mathcal{O}$ preserves limits — it sends colimits in **Set** to limits in **Com** k-**Alg**. Now let $\alpha : X \to Y$ be a surjective morphism in **Set**. Then, we can form the pullback square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow[\alpha]{} & Y \end{array}$$

Since **Set** is finitely cocomplete then we have the coequalizer $coeq(\pi_1, \pi_2) : X \to C$. By computing this coequalizer explicitly, we in fact have $C \cong Y$. By applying the functor $\mathcal{O}$, we obtain the equalizer

$$eq(\mathcal{O}(\pi_1), \mathcal{O}(\pi_2)) : \mathcal{O}(C) \to \mathcal{O}(X)$$

and the isomorphism of commutative k-algebras $\mathcal{O}(C) \cong \mathcal{O}(Y)$. This means that functions on Y can be written in terms of functions on X satisfying a "descent condition" — namely, their two pullbacks to $\mathcal{O}(X \times_Y X)$ via $\mathcal{O}(\pi_1)$ and $\mathcal{O}(\pi_2)$ are equal.

Descent theory generalises the above example by replacing morphisms of sets with morphisms of some "geometric objects" and by replacing k-vector spaces such as $\mathcal{O}(X)$ with certain categories associated to the geometric objects.

Let us examine Galois descent, which concerns vector spaces over fields. Let $k \subsetneq L$ be an extension of fields. Then there is an adjoint pair of functors

$$(Ind, Res)$$

where $Ind : k\text{-}\mathbf{Vect} \to L\text{-}\mathbf{Vect}$ sends a k-vector space V to the L-vector space $V \otimes_k L$ and $Res : L\text{-}\mathbf{Vect} \to k\text{-}\mathbf{Vect}$ is simply the restriction functor. This adjoint pair is actually a special case of the Hom-tensor adjunction in Example 2.1.3.

How can we apply Theorem 6.3.4 to this situation? The categories k-**Vect** and L-**Vect** are both finitely complete and finitely cocomplete and the functor Res is obviously conservative. So if we want to apply Theorem 6.3.4 then we have to show that Ind is a conservative functor. Since Res is conservative then it suffices to prove that the composite $Res \circ Ind$ is conservative.

To this end, assume that $f : V \to W$ is a morphism of k-vector spaces which induces an isomorphism $V \otimes_k L \to W \otimes_k L$ is a isomorphism of k-vector spaces. Since $k \subsetneq L$ is an extension of fields then $L \cong \bigoplus_i k$ as a k-vector space. Therefore the map

$$\bigoplus_i f : \bigoplus_i V \cong V \otimes_k L \to \bigoplus_i W \cong W \otimes_k L$$

is an isomorphism and thus, f is an isomorphism of k-vector spaces. We conclude that $Res \circ Ind$ and Ind are conservative functors.

Now an equalizer of a pair (f, g) in k-**Vect** is simply the kernel $\ker(f - g)$. From the fact that $L \cong \bigoplus_i k$ as k-vector spaces, the functor Ind preserves kernels and thus, preserves equalizers. At this point, Ind satisfies the conditions of Theorem 6.3.4.

So by the Barr-Beck theorem, Ind is a comonadic functor. Let us unpack what this means explicitly. The functor $S = Ind \circ Res$ is a comonad over L-**Vect** and the functor

$$\begin{array}{rccc} Ind_{enh}: & k\textbf{-Vect} & \to & CoAlg_S(L\textbf{-Vect}) \\ & Z & \mapsto & (Z \otimes_k L, Ind(\eta_Z)). \end{array}$$

(see equation (6.3)) defines an equivalence where $\eta : id_{L\textbf{-Vect}} \Rightarrow Ind \circ Res$ is the unit of adjunction for (Ind, Res). To put it simply, the category of k-vector spaces is equivalent to the category of S-coalgebras in the category of L-vector spaces. However, observe that the functor $S = Ind \circ Res$ sends a L-vector space V to

$$S(V) = V \otimes_k L \cong V \otimes_L (L \otimes_k L).$$

So the comonad S is simply given by tensoring with the L-coalgebra $L \otimes_k L$. By definition of a coalgebra, we deduce that k-**Vect** is equivalent to the category of $L \otimes_k L$-comodules in L-**Vect**. To be clear, the coalgebra strucutre on L is given by the coproduct

$$\begin{array}{rcc} L \otimes_k L & \to & (L \otimes_k L) \otimes_L (L \otimes_k L) \cong L \otimes_k L \otimes_k \\ l_1 \otimes l_2 & \mapsto & l_1 \otimes 1 \otimes l_2 \end{array}$$

and the counit

$$\begin{array}{rcc} L \otimes_k L & \to & L \\ l_1 \otimes l_2 & \mapsto & l_1 l_2. \end{array}$$

Now let us understand descent in the context of commutative rings. Let R and S be commutative rings and $\phi : R \to S$ is a morphism of commutative rings. Similarly to the Hom-tensor adjunction in Example 2.1.3, the functor

$$Ind = S \otimes_R (-) : R\textbf{-Mod} \to S\textbf{-Mod}$$

has a right adjoint given by the restriction functor

$$Res : S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}.$$

So by Theorem 5.1.1, Ind preserves colimits. Note that in a category of modules, finite products and finite coproducts coincide (direct sum). Thus, the functor Ind also preserves finite coproducts.

The specific type of descent we will investigate is given by the following definition.

Definition 6.4.1. Let R and S be commutative rings and $f : R \to S$ be a morphism of commutative rings. We say that f is **faithfully flat** if the functor $S \otimes_R (-)$ is faithful and preserves equalizers.

We claim that if $f : R \to S$ is faithfully flat then the tensor product functor $Ind = S \otimes_R (-)$ is comonadic. So assume that $f : R \to S$ is a faithfully flat morphism. By Theorem 6.3.4, it suffices to show that Ind is a conservative functor.

Assume that $\alpha : M \to N$ is a morphism in R-**Mod** and that

$$Ind(\alpha) : S \otimes_R M \to S \otimes_R N$$

is an isomorphism of S-modules. As proved in [Wei94], the coequalizer of α and the zero morphism $0_{M,N} : M \to N$ is simply the cokernel of α. Dually, the equalizer of α and $0_{M,N}$ is simply the kernel of α. Since Ind is faithfully flat then it must preserve both kernels and cokernels. Therefore, in order to prove Ind is conservative, it suffices to show that if P is a R-module and $Ind(P) \cong 0$ then $P \cong 0$.

So assume that P is a R-module and $Ind(P) \cong 0$. If Q is another R-module then there is an injective map

$$Hom_R(Q, P) \to Hom_S(Ind(Q), Ind(P)) \cong \{0\}$$

by faithfulness of Ind. Consequently, there is a unique morphism from Q to P and P is the final object in R-**Mod**. So P is isomorphic to the zero object and Ind is conservative.

Now by Theorem 6.3.4, Ind is a comonadic functor. The comonad $Ind \circ Res$ is explicitly the functor

$$M \mapsto S \otimes_R M \cong S \otimes_R S \otimes_S M.$$

Again by definition of a coalgebra, we obtain the following result.

Theorem 6.4.1. *Let $f : R \to S$ be a faithfully flat morphism of commutative rings. Then, the category R-**Mod** is equivalent to the the category of S-modules M equipped with a coassociative coaction morphism $M \to S \otimes_R M$ (this is just the category of S-coalgebras over S-**Mod**).*

So far, we have showed that if $f : R \to S$ is faithfully flat then the tensor product functor Ind is comonadic. This raises the question: Is there a necessary and sufficient condition for Ind to be a comonadic functor. It turns out that this is provided by the notion of a *pure monomorphism*.

Definition 6.4.2. Let R be a commutative ring and $\phi : M \to N$ be a morphism of R-modules. We say that ϕ is a **pure monomorphism** if the following statement is satisfied: If P is a R-module then the induced morphism $P \otimes_R M \to P \otimes_R N$ is injective.

Let $f : R \to S$ be a faithfully flat morphism of rings. To see that f is a pure monomorphism of R-modules, we have to show that if N is a R-module then the induced morphism $N \cong N \otimes_R R \to N \otimes_R S$ is injective. Now since f is faithfully flat, it suffices to show that the morphism of tensor products

$$\begin{array}{rccc} t: & N \otimes_R S & \to & (N \otimes_R S) \otimes_R S \\ & n \otimes s & \mapsto & n \otimes s \otimes 1_S. \end{array}$$

is injective. Consider the map

$$\begin{array}{rccc} m: & N \otimes_R S \otimes_R S & \to & N \otimes_R S \\ & n \otimes s_1 \otimes s_2 & \mapsto & n \otimes s_1 s_2. \end{array}$$

Then $m \circ t = id_{N \otimes_R S}$, t is a split monomorphism and thus injective. Thus $f : R \to S$ is a pure monomorphism.

Theorem 6.4.2. *Let R and S be commutative rings and $f : R \to S$ be a morphism of commutative rings. Then the tensor product functor $S \otimes_R (-) : R$-**Mod** $\to S$-**Mod** is comonadic if and only if f is a pure monomorphism.*

This theorem is due to [JT04, Corollary 5.4].

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