C2.7 Broadening Project: An introduction to protomodular categories

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September 15, 2024

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1 Motivating protomodular categories

Introduced by Bourn in [Bou91], the notion of a protomodular category is an abstraction stemming from the category of groups **Grp**. As explained in [Bou17, Section 4.2], pointed protomodular categories share some of the important properties of **Grp**. Namely,

(P1) Monomorphisms in a pointed protomodular category are precisely the morphisms with trivial kernel.

(P2) *Regular epimorphisms* in a pointed protomodular category are cokernels of their kernels.

(P3) Pointed protomodular categories possess *normal subobjects*, which generalise the notion of a normal subgroup.

(P4) A reflexive relation in a pointed protomodular category is an internal equivalence relation (see [BB04, Section A.2]). To put it succinctly, pointed protomodular categories are *Mal'cev* categories (see [BB04, Section 2.2]).

(P5) Pointed protomodular categories possess a specific class of objects termed *commutative objects*. Commutative objects generalise the notion of an abelian group.

The scope of this project is to motivate and define protomodular categories, prove various different characterisations of pointed protomodular categories and to show that properties (P1) and (P2) hold in pointed protomodular categories.

We begin with the well-known characterisation of split short exact sequences in the category of abelian groups **Ab**.

Definition 1.1. Let $\mathscr{C}$ be a category and A, B, C, D be objects in $\mathscr{C}$. Let $m : A \to B$ be a monomorphism and $f : C \to D$ be an epimorphism. We denote by id_A the identity map on A.

We say that f is a **split epimorphism** if there exists a morphism $s : D \to C$ such that $f \circ s = id_D$. We call the morphism s a **section** of f. Dually, we say that m is a **split monomorphism** if there exists a morphism $n : B \to A$ such that $n \circ m = id_A$.

Theorem 1.1. *Suppose that we have the following short exact sequence of abelian groups:*

$$0 \longrightarrow H \xrightarrow{f} G \xrightarrow{g} K \longrightarrow 0$$

Then, the following are equivalent:

1. *f is a split monomorphism in **Ab**,*
2. *g is a split epimorphism in **Ab**,*
3. *There exists an isomorphism $\theta : G \to H \oplus K$ such that the following diagram in **Ab** commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{f} & G & \xrightarrow{g} & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_H} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id_K} & & \\ 0 & \longrightarrow & H & \hookrightarrow & H \oplus K & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

The group morphism $H \oplus K \to K$ is projection onto the second direct summand.

A proof of Theorem 1.1 is given in [HS97, Lemma 4.6]. Note that the monomorphisms and epimorphisms in **Grp** are simply the injective and surjective group morphisms respectively (see [Lei14, Example 5.1.31, Example 5.2.19]). The point we want to accentuate here is that Theorem 1.1 fails in the category **Grp**.

Example 1.1. Let S_3 be the symmetric group and $A_3 \leq S_3$ denote the alternating group. The groups S_3 and A_3 fit into the short exact sequence

$$0 \longrightarrow A_3 \hookrightarrow S_3 \xrightarrow{\text{sgn}} \{\pm 1\} \longrightarrow 0.$$

Explicitly, sgn is the group homomorphism

$$\begin{array}{rccc} \text{sgn}: & S_3 & \to & \{\pm 1\} \\ & \sigma & \mapsto & (-1)^{|\{(i,j)\in\{1,2,3\}\times\{1,2,3\} \mid i<j,\ \sigma(i)>\sigma(j)\}|}. \end{array}$$

For instance,

$$\text{sgn}((12)) = (-1)^{|\{(1,2)\}|} = -1 \qquad \text{and} \qquad \text{sgn}((123)) = (-1)^{|\{(1,2),(1,3)\}|} = 1.$$

Observe that $S_3 \ncong A_3 \oplus \{\pm 1\}$ because $A_3 \oplus \{\pm 1\}$ is abelian whereas S_3 is not abelian. However, if we define the group homomorphism s by

$$\begin{array}{rccc} s: & \{\pm 1\} & \to & S_3 \\ & -1 & \mapsto & (12) \\ & 1 & \mapsto & (123) \end{array}$$

then $\text{sgn} \circ s = id_{\{\pm 1\}}$ and sgn is a split epimorphism in **Grp**.

In the category **Grp**, the presence of a split epimorphism instead yields [Bou17, Observation (D), Section 3.6] which we highlight below.

Theorem 1.2. *Let $f : G \to H$ be a split epimorphism in the category* ***Grp*** *and $s : H \to G$ be a morphism satisfying $f \circ s = id_H$. Let $K \leq G$ be a subgroup of G. If $\ker f \subseteq K$ and $s(H) \subseteq K$ then $K = G$.*

Proof. Assume that $f : G \to H$ is a split epimorphism in **Grp** and $s : H \to G$ is its section. Let K be a subgroup of G such that $\ker f \subseteq K$ and $s(H) \subseteq K$.

To show: (a) $G \subseteq K$.

(a) Assume that $g \in G$. Then

$$g = (s \circ f)(g)\big((s \circ f)(g^{-1})g\big).$$

Observe that $(s \circ f)(g) \in s(H) \subseteq K$ and $(s \circ f)(g^{-1})g \in \ker f \subseteq K$. Therefore, $g \in K$ and $G \subseteq K$. By part (a), we deduce that $G = K$ as required. □

Rephrased in the language of posets, the conclusion of Theorem 1.2 states that $G = s(H) \bigvee \ker f$. That is, G is the *join* of $\ker f$ and $s(H)$ (see [SS15, Section 2.4]). Compare this with Theorem 1.1 which says that in the category **Ab**, $G \cong H \oplus \ker f$.

The notion of a protomodular category stems from generalising Theorem 1.2 to pointed, finitely complete categories. Before proceeding, let us fix some notation. The zero object in a pointed category will be denoted by $*$. If A is an object in a pointed category then $\alpha_A : * \to A$ and $\tau_A : A \to *$ are the initial and terminal morphisms on A respectively. If B, C are objects in a pointed category then the composite $\alpha_C \circ \tau_B : B \to C$ is called the zero map between B and C. We denote this morphism by $0_{B,C}$.

Definition 1.2. Let $\mathscr{C}$ be a pointed, finitely complete category and $f : X \to Y$ be a morphism in $\mathscr{C}$. The **kernel** of f is a morphism $k_f : \ker f \to X$ in $\mathscr{C}$ such that the following commutative square in $\mathscr{C}$ is a pullback square:

$$\begin{array}{ccc} \ker f & \xrightarrow{k_f} & X \\ {\scriptstyle \tau_{\ker f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ * & \xrightarrow[\alpha_Y]{} & Y \end{array}$$

Generally in a pointed finitely complete category, the kernel of a morphism $f : X \to Y$ is defined as the equalizer of the pair $(f, 0_{X,Y})$. This definition is equivalent to Definition 1.2. Moreover in the category **Grp**, we recover the usual notion of the kernel of a group morphism. That is, if $f : G \to H$ is a group morphism then

$$\ker f = \{g \in G \mid f(g) = 1\}$$

and k_f is the inclusion $\ker f \hookrightarrow G$. See [Rie17, Example 3.1.14].

Definition 1.3. Let $\mathscr{C}$ be a category and $f : Y \to X$ and $g : Z \to X$ be morphisms in $\mathscr{C}$. We say that the pair (f, g) is **jointly extremally epic** if the following statement is satisfied: If $m : A \to X$ is a monomorphism which induces factorisations in the commutative diagram

$$\begin{array}{ccccc} & \nearrow & A & \nwarrow & \\ & & \big\downarrow{\scriptstyle m} & & \\ Y & \xrightarrow[f]{} & X & \xleftarrow[g]{} & Z \end{array}$$

then m is an isomorphism.

Using jointly extremally epic pairs of morphisms, we now rephrase Theorem 1.2 with category theory.

Theorem 1.3. *Let $f : G \to H$ be a split epimorphism in the category **Grp** and $s : H \to G$ be a morphism satisfying $f \circ s = id_H$. Let $k_f : \ker f \to G$ be the kernel of f. Then, the pair (s, k_f) is jointly extremally epic.*

Proof. Assume that $f : G \to H$ is a split epimorphism in **Grp** with section $s : H \to G$. Assume that $k_f : \ker f \to G$ is the kernel of f. Suppose that $m : A \to G$ is a group monomorphism which induces two factorisations making the following diagram in **Grp** commute:

$$\begin{array}{ccccc} & \overset{\gamma}{\nearrow} & A & \overset{\kappa}{\nwarrow} & \\ & & \big\downarrow{\scriptstyle m} & & \\ H & \xrightarrow[s]{} & G & \xleftarrow[k_f]{} & \ker f \end{array}$$

To be clear, m satisfies $m \circ \gamma = s$ and $m \circ \kappa = k_f$. Since m is a monomorphism in **Grp**, it is injective. In order to show that m is an isomorphism, it suffices to show that m is surjective. To this end, assume that $g \in G$. Then

$$g = (s \circ f)(g)\big((s \circ f)(g^{-1})g\big).$$

where $(s \circ f)(g) \in s(H)$ and $(s \circ f)(g^{-1})g \in \ker f$. The element

$$\gamma(f(g))\kappa\big((s \circ f)(g^{-1})g\big) \in A$$

satisfies

$$\begin{aligned} m\big(\gamma(f(g))\kappa(s(f(g^{-1}))g)\big) &= (m \circ \gamma)(f(g))(m \circ \kappa)(s(f(g^{-1}))g) \\ &= s(f(g))k_f(s(f(g^{-1}))g) \\ &= s(f(g))s(f(g^{-1}))g = g. \end{aligned}$$

In the second last equality, we used the fact that the kernel k_f of the group morphism f is the inclusion morphism $\ker f \hookrightarrow G$. So, m is surjective and consequently, m is a group isomorphism. We conclude that the pair of morphisms (s, k_f) is jointly extremally epic as required. □

2 Background material

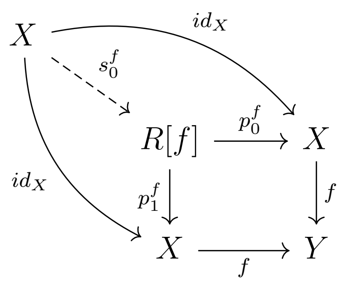
2.1 Kernel equivalence relations

In this section, we collate background material which will prove useful in later sections. In particular, the material in this section is needed to accurately state our promised characterisation of pointed protomodular categories in Theorem 3.1. Firstly, we need the following construction in a finitely complete category.

Definition 2.1. Let $\mathscr{C}$ be a finitely complete category and $f : X \to Y$ be a morphism in $\mathscr{C}$. Form the following pullback square:

$$\begin{array}{ccc} R[f] & \xrightarrow{p_0^f} & X \\ {\scriptstyle p_1^f}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{1}$$

By the universal property of the pullback, there exists a unique morphism $s_0^f : X \to R[f]$ making the following diagram commute:



The triple of morphisms (p_0^f, p_1^f, s_0^f) is called the **kernel equivalence relation** of f.

The kernel equivalence relation of a monomorphism is particularly nice.

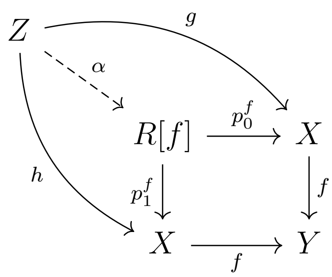
Theorem 2.1. *Let $\mathscr{C}$ be a finitely complete category and $f : X \to Y$ be a morphism in $\mathscr{C}$. Let (p_0^f, p_1^f, s_0^f) be the kernel equivalence relation on f. Then, f is a monomorphism if and only if $s_0^f : X \to R[f]$ is an isomorphism.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category, $f : X \to Y$ is a morphism in $\mathscr{C}$ and (p_0^f, p_1^f, s_0^f) is the kernel equivalence relation on f. First assume that f is a monomorphism. Since monomorphisms are stable under pullbacks, $p_0^f : R[f] \to X$ is also a monomorphism. So

$$p_0^f \circ (s_0^f \circ p_0^f) = id_X \circ p_0^f = p_0^f \circ id_{R[f]}$$

and subsequently $s_0^f \circ p_0^f = id_{R[f]}$. Since s_0^f is also a right inverse to p_0^f by construction of the kernel equivalence relation then s_0^f must be an isomorphism.

Conversely, assume that s_0^f is an isomorphism. Assume that $g, h : Z \to X$ are morphisms in $\mathscr{C}$ satisfying $f \circ g = f \circ h$. By using the universal property of the pullback square in diagram (1), there exists a unique morphism $\alpha : Z \to R[f]$ making the following diagram commute:



Since $s_0^f : X \to R[f]$ is an isomorphism and $p_0^f \circ s_0^f = p_1^f \circ s_0^f = id_X$ then $p_0^f = p_1^f$. By commutativity of the above diagram, we have $g = h$. So f is a monomorphism in $\mathscr{C}$. □

2.2 Conservative functors

A conservative functor is a functor which reflects isomorphisms.

Definition 2.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F is **conservative** if the following statement is satisfied: If f is a morphism in $\mathscr{C}$ and $F(f)$ is an isomorphism in $\mathscr{D}$ then f is an isomorphism.

In the proof of Theorem 3.1, we will make use of the following theorem stated in [Bou17, Exercise 2.1.6].

Theorem 2.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be finitely complete categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor which preserves finite limits. Then, F is conservative if and only if F is conservative on monomorphisms — that is, if f is a monomorphism in $\mathscr{C}$ and $F(f)$ is an isomorphism in $\mathscr{D}$ then f is an isomorphism.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are finitely complete categories and F is a functor which preserves finite limits. It suffices to show that if F is conservative on monomorphisms then F is conservative.

To this end, assume that F is conservative on monomorphisms and that $f : X \to Y$ is a morphism in $\mathscr{C}$ such that $F(f)$ is an isomorphism in $\mathscr{D}$. Let (p_0^f, p_1^f, s_0^f) denote the kernel equivalence relation on f (see Definition 2.1). By applying F to diagram (1), we obtain the following commutative square in $\mathscr{D}$:

$$\begin{array}{ccc} F(R[f]) & \xrightarrow{F(p_0^f)} & F(X) \\ {\scriptstyle F(p_1^f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array} \tag{2}$$

Since F preserves finite limits then the commutative square in diagram (2) is a pullback square. More precisely, it is the kernel equivalence relation on $F(f)$. Since $F(f)$ is an isomorphism then by Theorem 2.1, $F(s_0^f)$ is an isomorphism in $\mathscr{D}$. Consequently s_0^f is an isomorphism in $\mathscr{C}$ because F is conservative on monomorphisms. By applying Theorem 2.1 again, we deduce that f is a monomorphism.

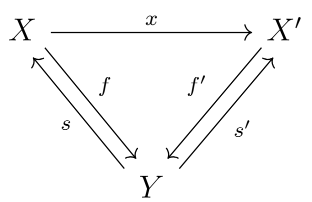
Now observe that f is a monomorphism in $\mathscr{C}$ such that $F(f)$ is an isomorphism in $\mathscr{D}$. Since F is conservative on monomorphisms then f is an isomorphism and subsequently, F is a conservative functor as required. □

2.3 Fibres above objects and base change functors

Following [Bou17, Section 1.6.5], we now define a category whose objects are split epimorphisms with a specific target object.

Definition 2.3. Let $\mathscr{C}$ be an arbitrary category and Y be an object in $\mathscr{C}$. The **fibre above** Y, denoted by $Pt_Y(\mathscr{C})$, is the category whose objects are split epimorphisms in $\mathscr{C}$ with target Y and whose morphisms are commutative triangles.

Specifically, if $f : X \to Y$ and $f' : X' \to Y$ are objects in $Pt_Y(\mathscr{C})$ with sections $s : Y \to X$ and $s' : Y \to X'$ respectively then a morphism from f to f' is a map $x : X \to X'$ which makes the following diagram in $\mathscr{C}$ commute:



That is, x satisfies $f = f' \circ x$ and $s' = x \circ s$ in $\mathscr{C}$. To be succinct, the object f in $Pt_Y(\mathscr{C})$ with accompanying section s will be denoted as the pair $(f, s) : X \leftrightarrow Y$.

We briefly remark that the category $Pt_Y(\mathscr{C})$ is a specific example of a fibre of a fibration, a concept outside of the scope of this project. See [Bor94b, Section 8.1] for a comprehensive introduction to fibrations. If $\mathscr{C}$ is a finitely complete category then a morphism $f : X \to Y$ in $\mathscr{C}$ induces a base change functor $f^* : Pt_Y(\mathscr{C}) \to Pt_X(\mathscr{C})$. In order to properly define this, we need the following lemma.

Lemma 2.3. *Let $\mathscr{C}$ be a finitely complete category and suppose we have the following pullback square in $\mathscr{C}$:*

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[y]{} & Y \end{array}$$

If f is a split epimorphism then p is also a split epimorphism.

Proof. Assume that we have the pullback square in $\mathscr{C}$ as stated in the lemma. Assume that $f : X \to Y$ is a split epimorphism and let $s : Y \to X$ be a section for f. By the universal property of the pullback, there exists a unique morphism $t : Z \to P$ such that the following diagram commutes:

$$\begin{array}{cccc} Z & & \xrightarrow{s \circ y} & \\ & \searrow^{t} & & \searrow \\ & P & \xrightarrow{q} & X \\ {\scriptstyle id_Z}\searrow & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ & Z & \xrightarrow[y]{} & Y \end{array}$$

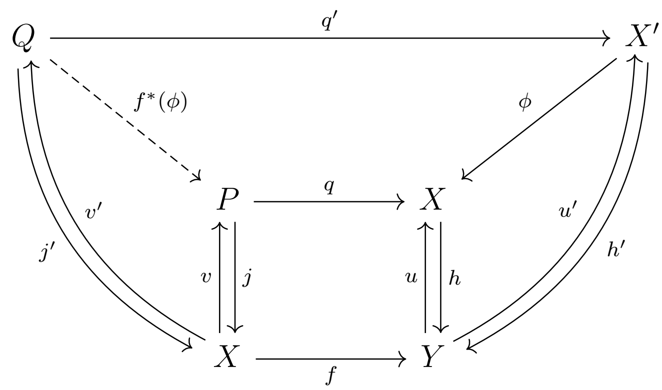
So $id_Z = p \circ t$ and hence, p is a split epimorphism as required. □

Definition 2.4. Let $\mathscr{C}$ be a finitely complete category and $f : X \to Y$ be a morphism in $\mathscr{C}$. The **base change functor** $f^* : Pt_Y(\mathscr{C}) \to Pt_X(\mathscr{C})$ is defined in the following manner:

If $g : Z \leftrightarrow Y$ is an object in $Pt_Y(\mathscr{C})$ then $f^*(g) : P \to X$ is a morphism in $\mathscr{C}$ defined by pulling back g along f. By Lemma 2.3, $f^*(g)$ is an object in $Pt_X(\mathscr{C})$ and f^* is well-defined on objects in $Pt_Y(\mathscr{C})$.

$$\begin{array}{ccc} P & \xrightarrow{q} & Z \\ {\scriptstyle f^*(g)}\downarrow\uparrow & & \uparrow\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Now let $(h, u) : X \leftrightarrow Y$ and $(h', u') : X' \leftrightarrow Y$ be objects in $Pt_Y(\mathscr{C})$. If $\phi : (h, u) \to (h', u')$ is a morphism in $Pt_Y(\mathscr{C})$, $f^*((h, u)) = (j, v)$ and $f^*((h', u')) = (j', v')$ then by using the universal property of the pullback, $f^*(\phi)$ is defined to be the unique morphism in $\mathscr{C}$ making the following diagram in $\mathscr{C}$ commute:



Since the LHS triangle in the above diagram commutes then $f^*(\phi)$ is a morphism in $Pt_X(\mathscr{C})$ from (j, v) to (j', v').

Obviously, there is some work required to show that base change functors are indeed functors. In order to not lengthen the document too much, we will omit the tedious details. Another fact we will use about fibres above objects is the following theorem.

Theorem 2.4. *Let $\mathscr{C}$ be a finitely complete category and X be an object in $\mathscr{C}$. Then the category $Pt_X(\mathscr{C})$ is also finitely complete. Moreover, if $f : X \to Y$ is a morphism in $\mathscr{C}$ then the base change functor $f^* : Pt_Y(\mathscr{C}) \to Pt_X(\mathscr{C})$ preserves finite limits.*

In [Bor94b, Proposition 8.5.2], Theorem 2.4 is proved in the more general context of fibred categories and fibrations.

3 Definition, characterisations and examples

Stated below is the most important definition in the document; that of a protomodular category. The definition originates from the proof of Theorem 1.3.

Definition 3.1. Let $\mathscr{C}$ be a pointed and finitely complete category. We say that $\mathscr{C}$ is **protomodular** if the following statement is satisfied: If $f : X \to Y$ is a split epimorphism in $\mathscr{C}$ with section $s : Y \to X$ and kernel map $k_f : \ker f \to X$ then the pair (s, k_f) is jointly extremally epic.

The main result of this section is Theorem 3.1 — a characterisation of pointed protomodular categories. All of these characterisations, except for one, are stated in [Bou17, Theorem 4.2.2]. The final characterisation of protomodular categories is in [BB04, Proposition 3.1.2] and can be thought of as a weakened form of the short five lemma which holds in an abelian category (see [Wei94, Exercise 1.3.3]).

Definition 3.2. Let $\mathscr{C}$ be a pointed finitely complete category. We say that the **split short five lemma** holds in $\mathscr{C}$ if the following statement is satisfied: Suppose we have the following commutative diagram in $\mathscr{C}$

$$\begin{array}{ccccccccc} * & \longrightarrow & \ker f & \xrightarrow{k_f} & B & \underset{s}{\overset{f}{\rightleftarrows}} & C & \longrightarrow & * \\ & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle c} & & \\ * & \longrightarrow & \ker g & \xrightarrow[k_g]{} & B' & \underset{t}{\overset{g}{\rightleftarrows}} & C' & \longrightarrow & * \end{array} \tag{3}$$

where f and g are split epimorphisms with sections s and t respectively. If a and c are isomorphisms then b is also an isomorphism.

We now state and prove the main theorem of the section.

Theorem 3.1. *Let $\mathscr{C}$ be a pointed finitely complete category. The following are equivalent:*

(C1) If $(f, s) : Y \leftrightarrow Z$ is a split epimorphism in $\mathscr{C}$, $y : X \to Z$ is a morphism in $\mathscr{C}$ and the downwards directed square

$$\begin{array}{ccc} P & \xrightarrow{x} & Y \\ \uparrow\downarrow & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ X & \xrightarrow[y]{} & Z \end{array}$$

is a pullback then the pair of morphisms (s, x) is jointly extremally epic.

(C2) If $y : X \to Z$ is a morphism in $\mathscr{C}$ then the base change functor $y^* : Pt_Z(\mathscr{C}) \to Pt_X(\mathscr{C})$ is conservative.

(C3) If T is an object in $\mathscr{C}$ and $\alpha_T : * \to T$ is the initial map then the base change functor $\alpha_T^* : Pt_T(\mathscr{C}) \to Pt_*(\mathscr{C})$ is conservative.

(C4) $\mathscr{C}$ is a protomodular category.

(C5) Suppose we have the following commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccccc} X & \xrightarrow{x} & X' & \xrightarrow{x'} & X'' \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\ Y & \xrightarrow[y]{} & Y' & \xrightarrow[y']{} & Y'' \end{array} \tag{4}$$

where f and f' are split epimorphisms with sections s and s' respectively. If the whole rectangle and the downwards directed LHS square are pullbacks then the RHS square is also a pullback.

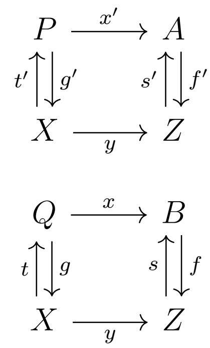
(C6) The split short five lemma holds in $\mathscr{C}$.

Proof. Assume that $\mathscr{C}$ is a pointed finitely complete category.

Conditions (C1) and (C2) are equivalent:

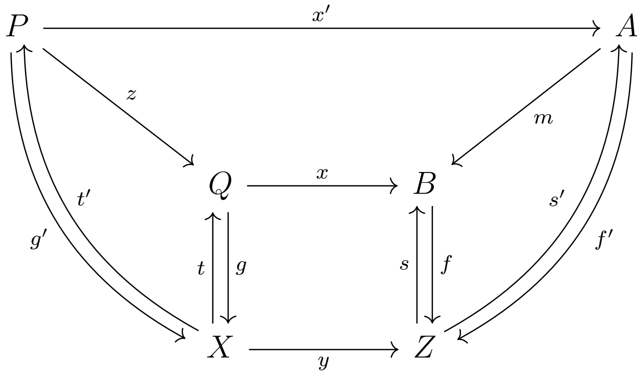
Assume that the condition (C1) holds. To see that condition (C2) holds, it suffices to show that if $y : X \to Z$ is a morphism in $\mathscr{C}$ then the base change functor $y^* : Pt_Z(\mathscr{C}) \to Pt_X(\mathscr{C})$ is conservative on monomorphisms by Theorem 2.4 and Theorem 2.2.

To this end, assume that $y : X \to Z$ is a morphism in $\mathscr{C}$. Assume that $(f', s') : A \leftrightarrow Z$ and $(f, s) : B \leftrightarrow Z$ are split eplimorphisms in $\mathscr{C}$ (objects in $Pt_Z(\mathscr{C})$) and that $m : (f', s') \to (f, s)$ is a monomorphism in $Pt_Z(\mathscr{C})$. Then $m : A \to B$ is a monomorphism in $\mathscr{C}$. Applying the functor y^*, we obtain pullback squares



(5)

and a monomorphism $z = y^*(m) : (g', t') \to (g, t)$. The two pullback squares fit into the following commutative diagram in $\mathscr{C}$ by definition of the base change functor (see Definition 2.4):



(6)

In particular, $z = y^*(m) : P \to Q$ is the unique morphism in $\mathscr{C}$ satisfying $m \circ x' = x \circ z$. Assume that z is an isomorphism in $Pt_X(\mathscr{C})$. By commutativity of diagram (6), $s = m \circ s'$ and

$$m \circ (x' \circ z^{-1}) = (m \circ x') \circ z^{-1} = (x \circ z) \circ z^{-1} = x.$$

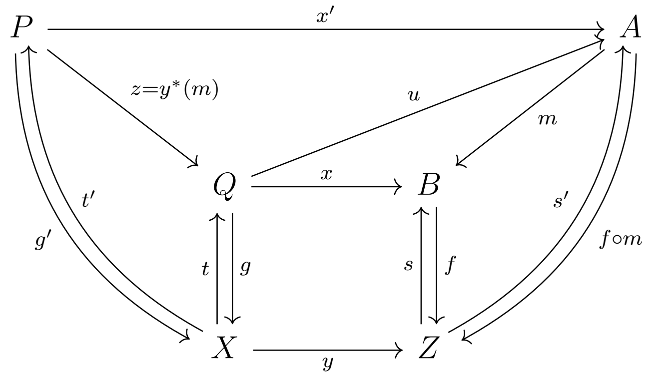
Since the pair (s, x) is jointly extremally epic by condition (C1) then m is an isomorphism in $\mathscr{C}$. Hence, $\mathscr{C}$ satisfies condition (C2).

Conversely, assume that condition (C2) holds so that if $y : X \to Y$ is a morphism in $\mathscr{C}$ then y^* is conservative. Let $(f, s) : X \leftrightarrow Y$ be a split epimorphism. Suppose that we obtain the pullback square in diagram (5) after taking the pullback of f along y.

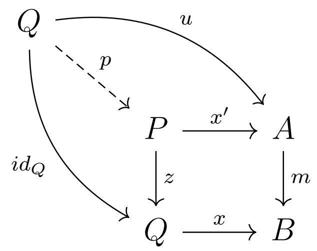
Now suppose that there exist a monomorphism $m : A \to B$ and morphisms $u : Q \to A$, $f' : Z \to A$ such that

$$s = m \circ s' \qquad \text{and} \qquad x = m \circ u.$$

Then m is a monomorphism in $Pt_Z(\mathscr{C})$ from (f, s) to $(f \circ m, s')$. By applying the base change functor y^* to (f, s), $(f \circ m, s')$ and m, we obtain the following commutative diagram:



Now the downwards directed square (with vertices Q, B, X, Z) and the downwards directed outer trapezium (with vertices P, A, X, Z) are both pullbacks. Therefore, the inner trapezium (with vertices P, A, Q, B) is also a pullback by [Bor94a, Proposition 2.5.9]. Consequently, there exists a unique morphism $p : Q \to P$ such that the following diagram in $\mathscr{C}$ commutes:



Since m is a monomorphism and monomorphisms are stable under pullbacks then z is also a monomorphism in $\mathscr{C}$. So

$$z \circ (p \circ z) = (z \circ p) \circ z = id_Q \circ z = z \circ id_P$$

and thus $p \circ z = id_P$. Therefore, $z = y^*(m)$ is an isomorphism and since y^* is conservative, m is an isomorphism. So, the pair (s, x) is jointly extremally epic as required.

Conditions (C2) and (C3) are equivalent:

It is clear that condition (C3) follows from condition (C2). Conversely, assume that if T is an object in $\mathscr{C}$ then the base change functor $\alpha_T^* : Pt_T(\mathscr{C}) \to Pt_1(\mathscr{C})$ is conservative. Assume that $y : X \to Z$ is a morphism in $\mathscr{C}$. Then $\alpha_Z = y \circ \alpha_X$ and we have the natural isomorphism

$$\alpha_Z^* = (y \circ \alpha_X)^* \simeq \alpha_X^* \circ y^*.$$

This is a natural isomorphism because taking pullbacks is associative (as explained in [BB04, Page 150]). The base change functors α_Z^* and α_X^* are both conservative by assumption. By [Bou17, Exercise 2.1.5], y^* must also be conservative and $\mathscr{C}$ satisfies condition (C2) as required.

Conditions (C3) and (C4) are equivalent:

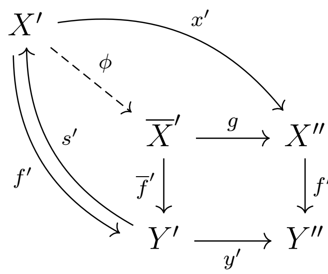
The equivalence of conditions (C3) and (C4) follows from exactly the same argument used to show the equivalence of conditions (C1) and (C2), with an initial map $\alpha_Z : * \to Z$ in place of the arbitrary morphism $y : X \to Z$.

Conditions (C2) and (C5) are equivalent:

Assume that $\mathscr{C}$ satisfies condition (C2). Assume that we have the commutative diagram in diagram (4) and that the whole rectangle and the downwards LHS square in diagram (4) are pullbacks. Taking the pullback of f'' along y' yields the pullback square

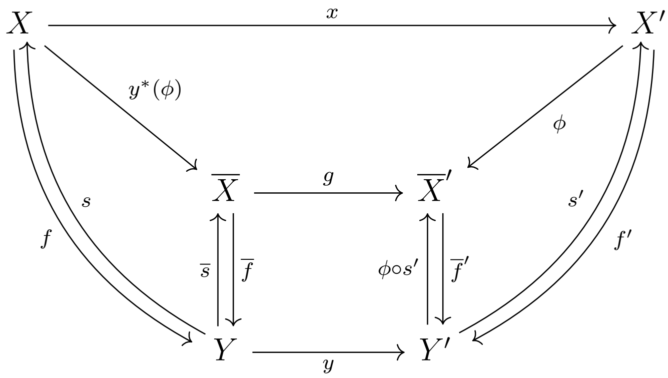
$$\begin{array}{ccc} \overline{X}' & \xrightarrow{g} & X'' \\ {\scriptstyle \overline{f}'}\downarrow & & \downarrow{\scriptstyle f''} \\ Y' & \xrightarrow[y']{} & Y'' \end{array}$$

Now there exists a unique morphism $\phi : X' \to \overline{X}'$ such that the following diagram commutes:



(7)

Now observe that ϕ is a morphism in $Pt_{Y'}(\mathscr{C})$ from (f', s') to $(\overline{f}', \phi \circ s')$. Applying the base change functor y^* to ϕ, we obtain the commutative diagram (as in Definition 2.4)



Now form the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\phi\circ x} & \overline{X}' & \xrightarrow{x'} & X'' \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle \phi\circ s'}\uparrow\downarrow{\scriptstyle \overline{f}'} & & \downarrow{\scriptstyle f''} \\ Y & \xrightarrow[y]{} & Y' & \xrightarrow[y']{} & Y'' \end{array}$$

The whole rectangle and the RHS square are both pullbacks by diagram (4) and by construction respectively. Therefore the LHS square is also a pullback. Subsequently,

$$y^*(\overline{f}', \phi \circ s') = (f, s) \qquad \text{and} \qquad y^*(\phi) = id_{(f,s)}.$$

Since $y^*(\phi)$ is an isomorphism then by condition (C2), ϕ is also an isomorphism and by diagram (7), the RHS square in diagram (4) is a pullback square.

Conversely, assume that condition (C5) is satisfied. Assume that $y : X \to Z$ is a morphism in $\mathscr{C}$. We want to show that the base change functor y^* is conservative. To this end, assume that $(f', s') : A \leftrightarrow Z$ and $(f, s) : B \leftrightarrow Z$ are split eplimorphisms in $\mathscr{C}$ (objects in $Pt_Z(\mathscr{C})$) and that $m : (f', s') \to (f, s)$ is a morphism in $Pt_Z(\mathscr{C})$. By applying y^* to m, we obtain diagram (6).

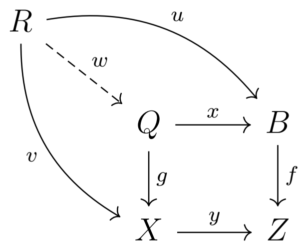
Assume that $z = y^*(m) : (g', t') \to (g, t)$ is an isomorphism in $Pt_X(\mathscr{C})$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} P & \xrightarrow{x'} & A & \xrightarrow{m} & B \\ {\scriptstyle t'}\uparrow\downarrow{\scriptstyle g'} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[y]{} & Z & \xrightarrow[id_Z]{} & Z \end{array} \tag{8}$$

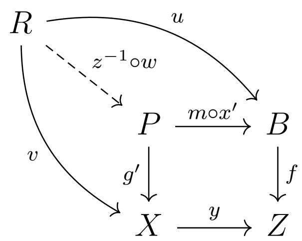
The downwards directed LHS square is a pullback. We claim that the outer square is also a pullback. Suppose that the following square in $\mathscr{C}$ commutes:

$$\begin{array}{ccc} R & \xrightarrow{u} & B \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{y} & Z \end{array}$$

Since the square with vertices Q, B, X, Z in diagram (6) is a pullback square, there exists a unique morphism $w : R \to Q$ such that the following diagram commutes:



Using the commutativity of diagram (6), we observe that $z^{-1} \circ w : R \to P$ is the unique morphism which makes the following diagram commute:



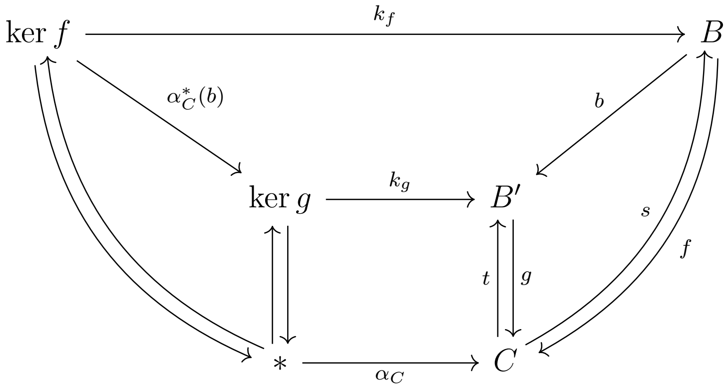
Therefore, the outer rectangle in diagram (8) is also a pullback. By assumption, we deduce that the RHS square in (8) is a pullback square. Since isomorphisms are stable under pullbacks and the identity map id_Z is an isomorphism then m must also be an isomorphism. So the base change functor y^* is conservative and condition (C2) holds.

Conditions (C3) and (C6) are equivalent:

First assume that condition (C6) holds; that is, the split short five lemma holds in $\mathscr{C}$. Assume that C is an object in $\mathscr{C}$. To see that α_C^* is conservative, assume that $b : (f, s) \to (g, t)$ is a morphism in $Pt_C(\mathscr{C})$. Then we have the following commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccccccccc} * & \longrightarrow & \ker f & \xrightarrow{k_f} & B & \underset{s}{\overset{f}{\rightleftarrows}} & C & \longrightarrow & * \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle id_C} & & \\ * & \longrightarrow & \ker g & \xrightarrow[k_g]{} & B' & \underset{t}{\overset{g}{\rightleftarrows}} & C & \longrightarrow & * \end{array} \tag{9}$$

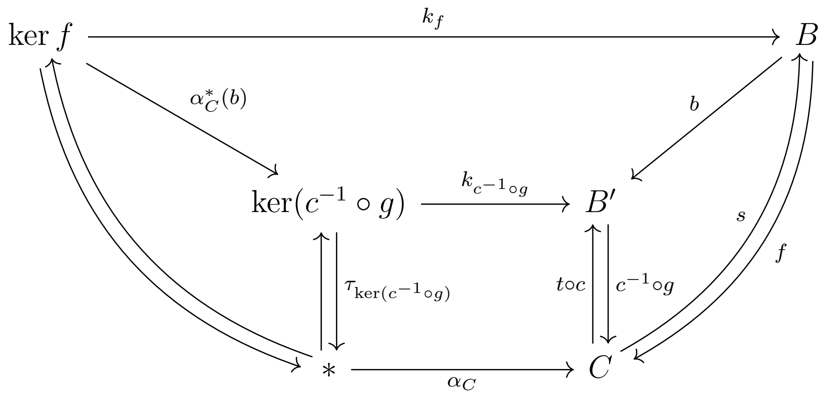
Applying the base change functor α_C^* to b, we obtain the following commutative diagram:



(10)

Assume that $\alpha_C^*(b) : \ker f \to \ker g$ is an isomorphism. Recall that $\alpha_C^*(b)$ is the unique morphism which makes the LHS triangle (with vertices $\ker f, \ker g, *$) and the upper trapezium (with vertices $\ker f, B, \ker g, B'$) commute. However, we also have $k_g \circ a = b \circ k_f$. So a also makes the upper trapezium in diagram (10) commute. By definition of the initial and terminal morphisms, a makes the LHS triangle in (10) commute. Thus by uniqueness, $\alpha_C^*(b) = a$ and a is an isomorphism. Since the split short five lemma holds in $\mathscr{C}$ then b must be an isomorphism by diagram (9). So α_C is a conservative functor and condition (C3) is satisfied.

Conversely, assume that condition (C3) holds and that we have diagram (3) in $\mathscr{C}$. Assume that a and c are both isomorphisms. We want to show that b is an isomorphism. Since c is an isomorphism then $c^{-1} \circ g : B' \to C$ is a split epimorphism with section $t \circ c : C \to B'$. So, b is a morphism in $Pt_C(\mathscr{C})$ from (f, s) to $(c^{-1} \circ g, t \circ c)$. Now apply the base change functor α_C^* to obtain the commutative diagram



(11)

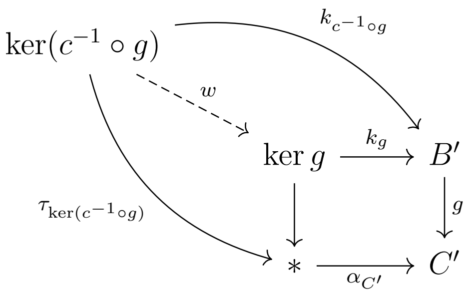
From diagrams (3) and (11),

$$k_g \circ a = b \circ k_f = k_{c^{-1} \circ g} \circ \alpha_C^*(b).$$

Since k_g and a are both monomorphisms (as a is an isomorphism) then $k_{c^{-1} \circ g} \circ \alpha_C^*(b)$ is a monomorphism. Since $k_{c^{-1} \circ g}$ is a monomorphism then $\alpha_C^*(b)$ is a monomorphism. Next observe that by commutativity of diagram (11),

$$c^{-1} \circ g \circ k_{c^{-1} \circ g} = \alpha_C \circ \tau_{\ker(c^{-1} \circ g)} = c^{-1} \circ \alpha_{C'} \circ \tau_{\ker(c^{-1} \circ g)}.$$

So $g \circ k_{c^{-1} \circ g} = \alpha_{C'} \circ \tau_{\ker(c^{-1} \circ g)}$ and by the universal property of the pullback, there exists a unique morphism $w : \ker(c^{-1} \circ g) \to \ker g$ such that the following diagram commutes:



The morphism $a^{-1} \circ w : \ker(c^{-1} \circ g) \to \ker f$ satisfies

$$\begin{aligned} k_{c^{-1} \circ g} \circ (\alpha_C^*(b) \circ a^{-1} \circ w) &= (k_{c^{-1} \circ g} \circ \alpha_C^*(b)) \circ a^{-1} \circ w \\ &= (b \circ k_f) \circ a^{-1} \circ w \qquad \text{(by diagram (11))} \\ &= k_g \circ a \circ a^{-1} \circ w \qquad \text{(by diagram (9))} \\ &= k_g \circ w = k_{c^{-1} \circ g}. \end{aligned}$$

Since $k_{c^{-1} \circ g}$ is a monomorphism then

$$\alpha_C^*(b) \circ (a^{-1} \circ w) = id_{\ker(c^{-1} \circ g)}.$$

Additionally, $\alpha_C^*(b) \circ (a^{-1} \circ w \circ \alpha_C^*(b)) = \alpha_C^*(b)$ and since $\alpha_C^*(b)$ is a monomorphism then

$$(a^{-1} \circ w) \circ \alpha_C^*(b) = id_{\ker f}.$$

Therefore $\alpha_C^*(b)$ is an isomorphism and since α_C^* is a conservative functor by condition (C3) then b is an isomorphism. So condition (C6) holds, thereby completing the proof. □

One important consequence of Theorem 3.1 is that conditions (C1), (C2) and (C5) do not use the assumption that $\mathscr{C}$ is pointed. Thus, we can define protomodularity in the case where $\mathscr{C}$ is not pointed.

Definition 3.3. Let $\mathscr{C}$ be a finitely complete category. We say that $\mathscr{C}$ is **protomodular** if any one of the equivalent conditions (C1), (C2) and (C5) hold in $\mathscr{C}$.

Theorem 3.1 and Definition 3.3 yield many examples of pointed and non-pointed protomodular categories.

Example 3.1. By Theorem 1.3, the category of groups **Grp** is the archetypal example of a (pointed) protomodular category. The subcategory of abelian groups **Ab** is also protomodular.

If $\mathscr{C}$ is a pointed finitely complete category then the functor category $\mathcal{F}(\mathscr{C}, \mathbf{Grp})$ and the internal category $\mathbf{Grp}(\mathscr{C})$ (see [Bor94a, Section 8]) are also protomodular. One can verify that these categories are protomodular by definition in a similar vein to Theorem 1.3 for **Grp**. In particular, if **Top** is the category of topological spaces then $\mathbf{Grp}(\mathbf{Top})$, the category of topological groups, is protomodular.

Example 3.2. If $\mathscr{A}$ is an abelian category then $\mathscr{A}$ is a protomodular category. This is because $\mathscr{A}$ satisfies the short five lemma and hence, the split short five lemma as stipulated by condition (C6). For instance, if R is a commutative ring then the category of R-modules **R-Mod** is protomodular.

Example 3.3. The dual of the category of sets $\mathbf{Set}^{op}$ and the category of unitary rings **URg** are examples of non-pointed protomodular categories, as stated in [Bou17, Page 46].

To provide more examples of protomodular categories, we will use the following theorem as stated in [BB04, Example 3.1.9].

Theorem 3.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be finitely complete categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor which preserves finite limits and is conservative. If $\mathscr{D}$ is protomodular then $\mathscr{C}$ is also protomodular.*

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are finitely complete categories. Assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor which preserves finite limits and is conservative. Assume that $\mathscr{D}$ is protomodular. We will show that condition (C1) holds in $\mathscr{C}$.

Suppose that $(f, s) : Y \leftrightarrow Z$ is a split epimorphism in $\mathscr{C}$ and that the downwards directed square in diagram (12) is a pullback in $\mathscr{C}$:

$$\begin{array}{ccc} P & \xrightarrow{x} & Y \\ \uparrow\downarrow & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ X & \xrightarrow[y]{} & Z \end{array} \tag{12}$$

We want to show that the pair (s, x) is jointly extremally epic. So assume that $m : Q \to Y$ is a monomorphism in $\mathscr{C}$ satisfying the following statement: there exist morphisms $u : P \to Q$ and $v : Z \to Q$ such that $m \circ u = x$ and $m \circ v = s$. By applying the functor F to diagram (12), we obtain a pullback square in $\mathscr{D}$ because F preserves finite limits. Since $\mathscr{D}$ is protomodular then the pair of morphisms $(F(s), F(x))$ in $\mathscr{D}$ is jointly extremally epic. Now $F(m)$ is a monomorphism satisfying $F(m) \circ F(u) = F(x)$ and $F(m) \circ F(v) = F(s)$. Therefore, $F(m)$ is an isomorphism and using the fact that F is conservative, we deduce that m is an isomorphism in $\mathscr{C}$. So the pair (s, x) is jointly extremally epic and $\mathscr{C}$ is protomodular by condition (C1). □

Example 3.4. By using Theorem 3.2, the following are true:

1. If $\mathscr{C}$ is an additive category with finite limits then it is protomodular. The proof uses the facts that the category $\mathrm{Add}(\mathscr{C}, \mathbf{Ab})$ of additive functors from

$\mathscr{C}$ to **Ab** is abelian (and hence protomodular) and that the Yoneda embedding

$$\begin{array}{rccc} Y: & \mathscr{C} & \to & \mathrm{Add}(\mathscr{C}^{op}, \mathbf{Ab}) \\ & X & \mapsto & Hom_{\mathscr{C}}(-, X) \end{array}$$

preserves finite limits and is conservative. See [BB04, Example 3.1.13].

2. If $\mathscr{C}$ is protomodular and Y is an object in $\mathscr{C}$ then the slice and coslice categories $\mathscr{C}/Y$ and $Y\backslash\mathscr{C}$ are both protomodular (see [BB04, Example 3.1.14]). In particular if G is a group then the slice category $\mathbf{Grp}/G$ is an example of a non-pointed protomodular category (see [Bou17, Example 4.2.6]).

Example 3.5. In the paper [Bou04], Bourn proved that the dual category of an elementary topos $\mathcal{E}$ (see [Bor94c, Chapter 5]) is protomodular.

4 Some consequences of protomodularity

4.1 Property (P1) — monomorphisms

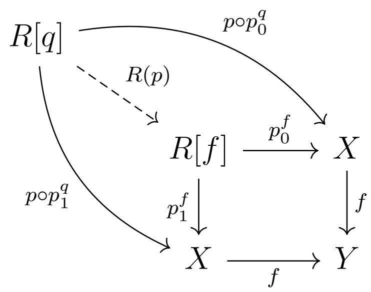
This section is dedicated to proving a few of the major consequences of a pointed finitely complete category being protomodular. The first one we will focus on is property (P1) — a characterisation of the monomorphisms in a pointed protomodular category.

In order to prove that property (P1) holds for a pointed protomodular category, we need to construct a particular pullback square involving kernel equivalence relations (see Definition 2.1).

Theorem 4.1. *Let $\mathscr{C}$ be a finitely complete category. Suppose that we have the commutative square*

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

in $\mathscr{C}$. Let (p_0^f, p_1^f, s_0^f) and (p_0^q, p_1^q, s_0^q) be the kernel equivalence relations of f and q respectively. Let $R(p) : R[q] \to R[f]$ be the unique morphism which makes the following diagram commute

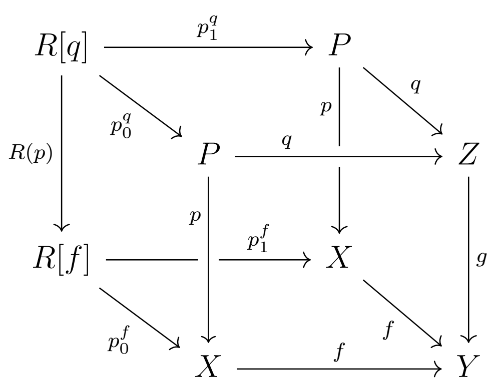


so that we obtain the following commutative diagram:

$$\begin{array}{ccccc} R[q] & \overset{p_0^q}{\underset{p_1^q}{\overrightarrow{\underleftarrow{\longrightarrow}}}} & P & \xrightarrow{q} & Z \\ {\scriptstyle R(p)}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ R[f] & \overset{p_0^f}{\underset{p_1^f}{\overrightarrow{\underleftarrow{\longrightarrow}}}} & X & \xrightarrow{f} & Y \end{array} \tag{13}$$

If the RHS square in diagram (13) *is a pullback square then the downwards directed LHS squares indexed by 0 and 1 are both pullback squares.*

Proof. Assume that $\mathscr{C}$ is a finitely complete category and that we have diagram (13). Assume that the RHS square in diagram (13) is a pullback square. The idea is to consider the following commutative diagram in $\mathscr{C}$:



(14)

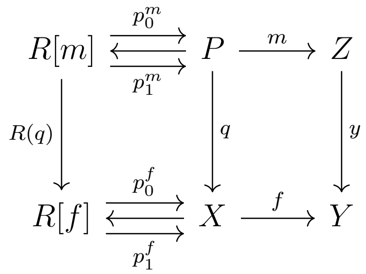
The top, bottom, front and right faces of the cube in diagram (14) are all pullback squares. By [Bou17, Corollary 1.6.3], the remaining two faces must also be pullback squares. □

Theorem 4.2. *Let $\mathscr{C}$ be a protomodular category. Suppose that we have the following pullback square in $\mathscr{C}$:*

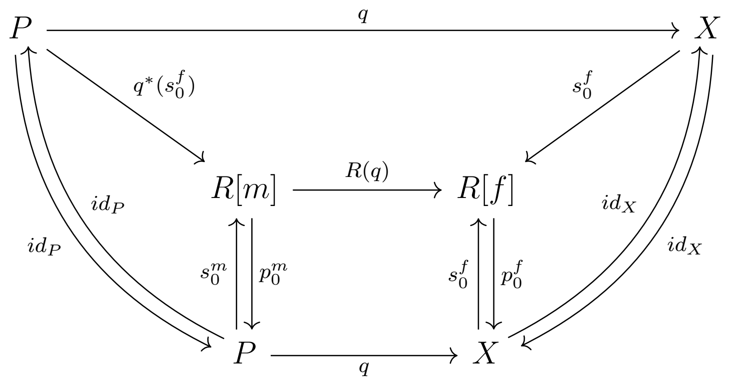
$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[y]{} & Y \end{array} \tag{15}$$

If m is a monomorphism then f is also a monomorphism.

Proof. Assume that $\mathscr{C}$ is a protomodular category. Suppose that we have the pullback square in diagram (15). Assume that p is a monomorphism. Let (p_0^f, p_1^f, s_0^f) and (p_0^m, p_1^m, s_0^m) be the kernel equivalence relations of f and m respectively. Form the commutative diagram



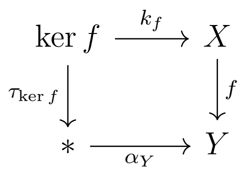
By Theorem 4.1, the LHS downwards directed squares indexed by 0 and 1 in the above diagram are pullback squares. Observe that s_0^f is a morphism in the category $Pt_X(\mathscr{C})$ from (id_X, id_X) to (p_0^f, s_0^f). The key idea is to use the definition of the base change functor from Definition 2.4 to compute the morphism $q^*(s_0^f)$. After some verification, we find that $q^*(s_0^f)$ is the unique morphism making the following diagram in $\mathscr{C}$ commute:



Notice that $s_0^m : P \to R[m]$ also makes the above diagram commute. By uniqueness, $q^*(s_0^f) = s_0^m$. Since m is a monomorphism then by Theorem 2.1, s_0^m is an isomorphism. By Definition 3.3, the base change functor q^* is conservative because $\mathscr{C}$ satisfies condition (C2). Therefore s_0^f is an isomorphism and by another application of Theorem 2.1, we deduce that f is a monomorphism as required. □

Corollary 4.3. *Let $\mathscr{C}$ be a pointed protomodular category and $f : X \to Y$ be a morphism in $\mathscr{C}$. Then f is a monomorphism if and only if* $\ker f \cong *$ *(the kernel of f is the zero object).*

Proof. Assume that $\mathscr{C}$ is a pointed protomodular category and that $f : X \to Y$ is a morphism in $\mathscr{C}$. First assume that f is a monomorphism. By Definition 1.2, we have the following pullback square in $\mathscr{C}$:



Since monomorphisms are stable under pullback then the terminal morphism $\tau_{\ker f}$ is a monomorphism. We also have $\tau_{\ker f} \circ \alpha_{\ker f} = id_*$. So

$$\tau_{\ker f} \circ (\alpha_{\ker f} \circ \tau_{\ker f}) = id_* \circ \tau_{\ker f} = \tau_{\ker f} \circ id_{\ker f}$$

and subsequently $\alpha_{\ker f} \circ \tau_{\ker f} = id_{\ker f}$. Therefore $\tau_{\ker f}$ is an isomorphism and $\ker f \cong *$.

Conversely assume that $\ker f \cong *$. Then the terminal morphism $\tau_{\ker f}$ is an isomorphism and thus a monomorphism. By Theorem 4.2, f must also be a monomorphism. We conclude that property (P1) holds for pointed protomodular categories. □

4.2 Property (P2) — regular epimorphisms

Recall the notion of a regular epimorphism from [Bor94a, Definition 4.3.1].

Definition 4.1. Let $\mathscr{C}$ be a category. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. We say that f is a **regular epimorphism** if there exists a pair of morphisms $g, h : Z \to X$ such that f is the coequalizer of the pair (g, h).

If $\mathscr{C}$ is a finitely complete category and $f : X \to Y$ is a morphism in $\mathscr{C}$ then it is not too difficult to show that the following statements are equivalent:

1. f is a regular epimorphism,
2. f is the coequalizer of the pair (p_0^f, p_1^f) where (p_0^f, p_1^f, s_0^f) is the kernel equivalence relation of f.

In what follows, we will use the second characterisation of regular epimorphisms in a finitely complete category. This particular characterisation appears in [Bou17, Definition 1.7.5].

Example 4.1. In the categories **Set**, **Grp** and **Ab**, the regular epimorphisms are exactly the surjective morphisms (see [Bor94a, Example 4.3.10.a]). Let $\mathbf{Ban}_1$ be the category whose objects are Banach spaces and whose morphisms are linear operators with operator norm less than or equal to 1. The regular epimorphisms in $\mathbf{Ban}_1$ turn out to be isometric, injective linear operators. See [Bor94a, Example 4.3.10.e].

In order to prove that property (P2) holds for a pointed protomodular category, we will make use of the following result which is proved by applying the definitions of a jointly extremally epic pair of morphisms and an equalizer.

Theorem 4.4. *Let $\mathscr{C}$ be a finitely complete category. Let $f : X \to Z$ and $f' : Y \to Z$ be morphisms in $\mathscr{C}$. Suppose that the pair (f, f') is jointly extremally epic. Then, (f, f') is jointly epic — namely it satisfies the following property: If $h, h' : Z \to T$ are morphisms in $\mathscr{C}$ then $h = h'$ if and only if $h \circ f = h' \circ f$ and $h \circ f' = h' \circ f'$.*

Theorem 4.5. *Let $\mathscr{C}$ be a protomodular category and $f : X \to Y$ be a morphism in $\mathscr{C}$. Suppose that we have the pullback square in $\mathscr{C}$:*

$$\begin{array}{ccc} P & \xrightarrow{p} & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{16}$$

If f is a regular epimorphism and p is an epimorphism then diagram (16) *is a pushout square from the top left corner.*

Proof. Assume that $\mathscr{C}$ is a protomodular category and that we have the pullback square in diagram (16). Assume that f is a regular epimorphism and that p is an epimorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc} R[p] & \overset{p_0^p}{\underset{p_1^p}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} & P & \xrightarrow{p} & Z \\ {\scriptstyle R(q)}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ R[f] & \overset{p_0^f}{\underset{p_1^f}{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}}} & X & \xrightarrow{f} & Y \end{array}$$

By Theorem 4.1, the downwards directed LHS square indexed by 0 is a pullback square. Since $\mathscr{C}$ is a protomodular category then it satisfies condition (C1) and so the pair of morphisms $(s_0^f, R(q))$ is jointly extremally epic. By Theorem 4.4, $(s_0^f, R(q))$ is jointly epic.

Now suppose that we have the following commutative square in $\mathscr{C}$:

$$\begin{array}{ccc} P & \xrightarrow{p} & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow[u]{} & T \end{array}$$

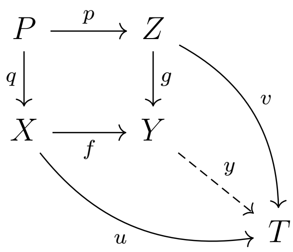
By direct computation, we have

$$(u \circ p_0^f) \circ s_0^f = u = (u \circ p_1^f) \circ s_0^f$$

and

$$\begin{aligned} (u \circ p_0^f) \circ R(q) &= (u \circ q) \circ p_0^p \\ &= (v \circ p) \circ p_0^p = (v \circ p) \circ p_1^p \\ &= (u \circ q) \circ p_1^p = (u \circ p_1^f) \circ R(q). \end{aligned}$$

Since the pair $(s_0^f, R(q))$ is jointly epic then $u \circ p_0^f = u \circ p_1^f$. Now f is a regular epimorphism and is thus the coequalizer of the pair (p_0^f, p_1^f). By the universal property of the coequalizer, there exists a unique morphism $y : Y \to T$ such that the following diagram commutes:



Note that the triangle with vertices Z, Y and T commutes because p is an epimorphism. So diagram (16) is a pushout from the top left corner. This completes the proof. □

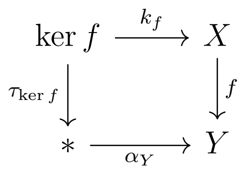
In the proof of Corollary 4.6, the notion of cokernel that we will use is simply the dual definition to Definition 1.2. In particular, in a pointed finitely complete category, the cokernel of a morphism $g : Y \to Z$ is the coequalizer of the pair $(g, 0_{Y,Z})$ where $0_{Y,Z} : Y \to Z$ is the zero map.

Corollary 4.6. *Let $\mathscr{C}$ be a pointed protomodular category and $f : X \to Y$ be a morphism in $\mathscr{C}$. Then f is a regular epimorphism if and only if*

$$f = coker(k_f : \ker f \to X).$$

Proof. Assume that $\mathscr{C}$ is a pointed protomodular category and $f : X \to Y$ is a morphism in $\mathscr{C}$. If f is the cokernel of its kernel map $k_f : \ker f \to X$ then f is the coequalizer of k_f and the zero map $0_{\ker f,X}$. By definition, f is a regular epimorphism.

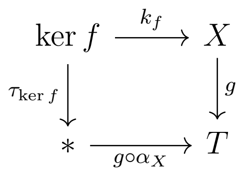
Conversely, assume that f is a regular epimorphism. By Theorem 4.5, the pullback square



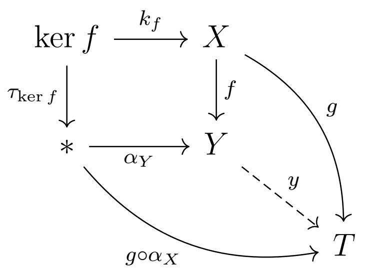
is also a pushout from the top left corner since the terminal map $\tau_{\ker f}$ is an epimorphism. Observe that

$$f \circ k_f = \alpha_Y \circ \tau_{\ker f} = 0_{\ker f,Y} = f \circ 0_{\ker f,X}.$$

Now assume that $g : X \to T$ satisfies $g \circ k_f = g \circ 0_{\ker f,X}$. Then the following square is commutative



and by the universal property of the pushout, there exists a unique morphism $y : Y \to T$ making the following diagram commute:



Hence, $f = \mathrm{coker}(k_f)$ and property (P2) holds in a pointed protomodular category. □

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