Eigenvalues of Minor Matrices

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Contents

Chapter 1

Introduction

1.1 The scenario

Let $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ be a Hermitian matrix such that

$$A = P\ diag[\lambda_1, \ldots, \lambda_n]\ P^{-1}. \tag{1.1}$$

where $P \in GL_n(\mathbb{C})$ is unitary ($P^{-1} = P^*$) and $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of A. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors. The eigenvector v_j is the j^{th} column of P.

Let $k \in \mathbb{Z}_{>0}$ such that $k \leq n$. Define $\mathbb{Z}_{[1,n]} = \{1, 2, \ldots, n\}$ and

$$T_{\binom{n}{k}} = \{S \subseteq \mathbb{Z}_{[1,n]} \mid |S| = k\}.$$

Here, $|S|$ refers to the cardinality of the set S. If $L = \{\ell_1, \ldots, \ell_k\}$ and $M = \{m_1, \ldots, m_k\}$ are elements of $T_{\binom{n}{k}}$ then let $A_{L,M}$ be the $k \times k$ matrix formed from rows $\ell_1, \ell_2, \ldots, \ell_k$ of A and columns $m_1, m_2, \ldots, m_k$ of A. Also, for $j \in \{1, 2, \ldots, n\}$ let $j^c = \mathbb{Z}_{[1,n]} - \{j\}$.

The question we would like to investigate is: Can we use equation (1.1) to gain information about the eigenvectors of the minor matrix $A_{1^c,1^c}$? The main reason for beginning with $A_{1^c,1^c}$ is because $A_{1^c,1^c} \in M_{(n-1)\times(n-1)}(\mathbb{C})$ is Hermitian. This is because A is Hermitian.

By the spectral theorem, there exists a unitary matrix $R \in GL_{n-1}(\mathbb{C})$ such that

$$R^{-1} A_{1^c,1^c} R = diag[\mu_1, \mu_2, \ldots, \mu_{n-1}]$$

where $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are the eigenvalues of $A_{1^c,1^c}$.

Define $Q \in GL_n(\mathbb{C})$ as the block matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}.$$

Due to the block structure on Q, we have

$$Q^{-1}AQ = \begin{pmatrix} a_{11} & * \\ * & diag[\mu_1, \ldots, \mu_{n-1}] \end{pmatrix}.$$

Here, $*$ denotes unnecessary elements. Let $B = Q^{-1}AQ$. Then,

$$B = (Q^{-1}P)diag[\lambda_1, \ldots, \lambda_n](P^{-1}Q).$$

If $j \in \{1, 2, \ldots, n\}$ then

$$\lambda_j I_n - B = (Q^{-1}P)D_j(P^{-1}Q). \tag{1.2}$$

where $D_j = diag[\lambda_j - \lambda_1, \ldots, \lambda_j - \lambda_n]$. Now apply Λ^{n-1} to both sides of equation (1.2) and take the $1^c, 1^c$ element. We obtain on the LHS

$$\left(\Lambda^{n-1}(\lambda_j I_n - B)\right)_{1^c,1^c} = \prod_{i=1}^{n-1}(\lambda_j - \mu_i). \tag{1.3}$$

The expression obtained on the RHS is more difficult to compute. Firstly, we have

$$\Lambda^{n-1}(D_j) = diag[0, \ldots, 0, \prod_{i=1, i\neq j}^{n}(\lambda_j - \lambda_i), 0, \ldots, 0]. \tag{1.4}$$

The non-zero term $\prod_{i=1,i\neq j}^{n}(\lambda_j - \lambda_i)$ is the j^c, j^c element of $\Lambda^{n-1}(D_j)$.

The bottom rows of $\Lambda^{n-1}(Q)$ and $\Lambda^{n-1}(Q^{-1})$ which are indexed by 1^c are respectively,

$$[0, 0, \ldots, 0, u] \qquad \text{and} \qquad [0, 0, \ldots, 0, u^{-1}]$$

where $u = \det(R)$ is a complex number of magnitude 1. Similarly, the rightmost columns of $\Lambda^{n-1}(Q)$ and $\Lambda^{n-1}(Q^{-1})$, which are again indexed by 1^c are respectively

$$[0, 0, \ldots, 0, u]^T \qquad \text{and} \qquad [0, 0, \ldots, 0, u^{-1}]^T$$

If $L = \{\ell_1, \dots, \ell_k\} \in T_{\binom{n}{k}}$ then we define $v_L = v_{\ell_1} \wedge \cdots \wedge v_{\ell_k}$. The v_L form the columns of $\Lambda^k(P)$. If $M \in T_{\binom{n}{k}}$ then $v_{L,M}$ denotes the M element of v_L. Now fix $r, s \in \{1, 2, \dots, n\}$. Then,

$$(\Lambda^{n-1}(Q^{-1}P))_{1^c,r^c} = \sum_{L \in T_{\binom{n}{n-1}}} (\Lambda^{n-1}(Q^{-1}))_{1^c,L}(\Lambda^{n-1}(P))_{L,r^c} = u^{-1}(\Lambda^{n-1}(P))_{1^c,r^c}$$

Hence,

$$(\Lambda^{n-1}(Q^{-1}P))_{1^c,r^c} = u^{-1}(\Lambda^{n-1}(P))_{1^c,r^c} = u^{-1}v_{r^c,1^c}. \tag{1.5}$$

By a similar computation, we find that

$$(\Lambda^{n-1}(P^{-1}Q))_{s^c,1^c} = u(\Lambda^{n-1}(P^{-1}))_{s^c,1^c} = u\overline{(\Lambda^{n-1}(P))_{1^c,s^c}} = u\overline{v_{s^c,1^c}}. \tag{1.6}$$

Finally, by applying Λ^{n-1} to the RHS of equation (1.2) and taking the $1^c, 1^c$ element, we obtain from equations (1.4), (1.5) and (1.6)

$$\prod_{i=1, i\neq j}^{n} (\lambda_j - \lambda_i)\ u\ \overline{v_{j^c,1^c}}\ u^{-1}\ v_{j^c,1^c} = \prod_{i=1, i\neq j}^{n} (\lambda_j - \lambda_i)\ |v_{j^c,1^c}|^2.$$

By equating with equation (1.8), we obtain

$$|v_{j^c,1^c}|^2 = \frac{\prod_{i=1}^{n-1}(\lambda_j - \mu_i)}{\prod_{i=1,i\neq j}^{n}(\lambda_j - \lambda_i)} \tag{1.7}$$

Notice that equation (1.7) is very similar to the *eigenvector-eigenvalue identity*, as applied to the matrix $A_{1^c,1^c}$. In fact, we will prove eigenvector-eigenvalue identity for the minor matrix $A_{1^c,1^c}$ in the next section. The eigenvector-eigenvalue identity appears in [DPTZ22, Theorem 1].

1.2 Proving the eigenvector-eigenvalue identity

The idea is to take Υ^{n-1} of both sides of equation (1.2) and then take the $1, 1$ element. Once we do this, the LHS becomes

$$\left(\Upsilon^{n-1}(\lambda_j I_n - B)\right)_{1,1} = \prod_{i=1}^{n-1}(\lambda_j - \mu_i). \tag{1.8}$$

In order to compute the resulting expression on the RHS, we require a few intermediate expressions. Since P and Q are invertible, we have

$$\Upsilon^{n-1}(P) = \det(P)P^{-1} = \det(P)P^{*}. \tag{1.9}$$

$$\Upsilon^{n-1}(P^{-1}) = \det(P)^{-1}P. \tag{1.10}$$

$$\Upsilon^{n-1}(Q) = \det(Q)Q^{-1}. \tag{1.11}$$

$$\Upsilon^{n-1}(Q^{-1}) = \det(Q)^{-1}Q. \tag{1.12}$$

We also have from the definition of Υ^{n-1}

$$(\Upsilon^{n-1}(D_j))_{\ell,\ell} = \begin{cases} \prod_{i=1,i\neq j}^{n}(\lambda_j - \lambda_i), & \text{if } \ell = j, \\ 0, & \text{if } \ell \neq j. \end{cases} \tag{1.13}$$

All the non-diagonal entries of $\Upsilon^{n-1}(D_j)$ are zero because $\Upsilon^{n-1}(D_j)$ is a diagonal matrix. We wish to compute the expression

$$(\Upsilon^{n-1}(Q^{-1}PD_jP^{-1}Q))_{1,1}.$$

Using equations (1.9), (1.10), (1.11), (1.12) and (1.13), we compute for $s \in \{1, 2, \dots, n\}$ the following expressions:

$$\begin{aligned}
(\Upsilon^{n-1}(P^{-1}Q))_{1,s} &= (\Upsilon^{n-1}(Q)\Upsilon^{n-1}(P^{-1}))_{1,s} \\
&= \sum_{r=1}^{n}(\Upsilon^{n-1}(Q))_{1,r}(\Upsilon^{n-1}(P^{-1}))_{r,s} \\
&= \sum_{r=1}^{n}(\det(Q)Q^{-1})_{1,r}(\det(P)^{-1}P)_{r,s} \\
&= \det(Q)\det(P)^{-1}\sum_{r=1}^{n}(Q^{-1})_{1,r}P_{r,s} \\
&= \det(Q)\det(P)^{-1}P_{1,s}.
\end{aligned}$$

$$
\begin{aligned}
(\Upsilon^{n-1}(Q^{-1}P))_{s,1} &= (\Upsilon^{n-1}(P)\Upsilon^{n-1}(Q^{-1}))_{s,1} \\
&= \sum_{r=1}^{n} (\Upsilon^{n-1}(P))_{s,r}(\Upsilon^{n-1}(Q^{-1}))_{r,1} \\
&= \sum_{r=1}^{n} (\det(P)P^*)_{s,r}(\det(Q)^{-1}Q)_{r,1} \\
&= \det(P)\det(Q)^{-1}\sum_{r=1}^{n}(P^*)_{s,r}Q_{r,1} \\
&= \det(P)\det(Q)^{-1}(P^*)_{s,1}.
\end{aligned}
$$

$$
\begin{aligned}
(\Upsilon^{n-1}(P^{-1}Q)\Upsilon^{n-1}(D_j))_{1,s} &= \sum_{r=1}^{n}(\Upsilon^{n-1}(P^{-1}Q))_{1,r}(\Upsilon^{n-1}(D_j))_{r,s} \\
&= (\Upsilon^{n-1}(P^{-1}Q))_{1,s}(\Upsilon^{n-1}(D_j))_{s,s} \\
&= (\Upsilon^{n-1}(P^{-1}Q))_{1,s}\prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i)\delta_{s,j} \\
&= \det(Q)\det(P)^{-1}\prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i)P_{1,s}\delta_{s,j}.
\end{aligned}
$$

The symbol $\delta_{s,j}$ is the Kronecker delta. Putting all these computations together, we have

$$
\begin{aligned}
(\Upsilon^{n-1}(Q^{-1}PD_jP^{-1}Q))_{1,1} &= (\Upsilon^{n-1}(P^{-1}Q)\Upsilon^{n-1}(D_j)\Upsilon^{n-1}(Q^{-1}P))_{1,1} \\
&= \sum_{s=1}^{n}(\Upsilon^{n-1}(P^{-1}Q)\Upsilon^{n-1}(D_j))_{1,s}(\Upsilon^{n-1}(Q^{-1}P))_{s,1} \\
&= \sum_{s=1}^{n}\Big(\det(Q)\det(P)^{-1}\prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i)P_{1,s}\delta_{s,j}\Big) \\
&\quad \Big(\det(P)\det(Q)^{-1}(P^*)_{s,1}\Big) \\
&= \prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i)\sum_{s=1}^{n}P_{1,s}(P^*)_{s,1}\delta_{s,j} \\
&= \prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i)P_{1,j}(P^*)_{j,1} \\
&= \prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i)v_{j,1}\overline{v_{j,1}} = |v_{j,1}|^2\prod_{i=1,i\neq j}^{n}(\lambda_j-\lambda_i).
\end{aligned}
$$

By equating the above equation with equation (1.8), we obtain

$$\prod_{i=1}^{n-1}(\lambda_j - \mu_i) = |v_{j,1}|^2 \prod_{i=1, i\neq j}^{n}(\lambda_j - \lambda_i)$$

and

$$|v_{j,1}|^2 = \frac{\prod_{i=1}^{n-1}(\lambda_j - \mu_i)}{\prod_{i=1,i\neq j}^{n}(\lambda_j - \lambda_i)} \tag{1.14}$$

which is the eigenvector-eigenvalue identity applied to the minor matrix $A_{1^c,1^c}$.

Are we able to extend equation (1.14) to the other $(n-1)\times(n-1)$ minor matrices of A? The answer is only partially. Assume that $u \in \mathbb{Z}_{[1,n]}$. Let $\mu_1^{(u)}, \mu_2^{(u)}, \ldots, \mu_{n-1}^{(u)}$ be the eigenvalues of the minor A_{u^c,u^c}. Let $w_u \in GL_n(\mathbb{C})$ be the permutation matrix such that the product $w_u A$ is obtained from A by swapping the first and u^{th} rows of A and the product Aw_u is obtained from A by swapping the first and u^{th} columns of A.

Since $w_u^2 = I_n$, we compute directly that

$$w_u A w_u = (w_u P w_u)(w_u diag[\lambda_1, \ldots, \lambda_n] w_u)(w_u P^{-1} w_u).$$

By applying equation (1.14) to $w_u A w_u$, we obtain

$$|v_{j,u}|^2 = \frac{\prod_{i=1}^{n-1}(\lambda_j - \mu_i^{(u)})}{\prod_{i=1,i\neq j}^{n}(\lambda_j - \lambda_i)}. \tag{1.15}$$

By repeating the argument outlined in the first section for $w_u A w_u$, we also find that

$$|v_{j^c,u^c}|^2 = \frac{\prod_{i=1}^{n-1}(\lambda_j - \mu_i^{(u)})}{\prod_{i=1,i\neq j}^{n}(\lambda_j - \lambda_i)} \tag{1.16}$$

We are unable to extend this argument to a minor matrix A_{u^c,v^c} with u and v distinct because A_{u^c,v^c} is not Hermitian in general.

1.3 Generalising the eigenvector-eigenvalue identity

In this section, we prove generalisations of equations (1.15) and (1.16). We will first state the generalisation of equation (1.15).

Theorem 1.3.1. *Let $n, k \in \mathbb{Z}_{>0}$ such that $k \leq n$. Let $A \in M_{n\times n}(\mathbb{C})$ satisfy $AA^* = A^*A$ and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. Let $v_1, v_2, \dots, v_n$ be the corresponding eigenvectors. If $L = \{l_1, l_2, \dots, l_k\} \in T_{\binom{n}{k}}$ then define*

$$v_L = v_{l_1} \wedge v_{l_2} \wedge \cdots \wedge v_{l_k}.$$

For $M \in T_{\binom{n}{k}}$, let $v_{L,M} \in \mathbb{C}$ be the M component of v_L. If $\tau \in \mathbb{C}$ and $M, N \in T_{\binom{n}{k}}$ then

$$\sum_{L\in T_{\binom{n}{k}}} \Big(\prod_{\ell\in L}(\tau - \lambda_\ell)\Big)\, v_{L,M}\, \overline{v_{L,N}} = \big(\Lambda^k(\tau I_n - A)\big)_{M,N}.$$

In particular, if $j \in \{1, 2, \dots, n\}$, $M = N = \{m_1, m_2, \dots, m_k\}$ and $A_{M,M}$ is the $k \times k$ matrix formed from rows $m_1, m_2, \dots, m_k$ and columns $m_1, m_2, \dots, m_k$ of A then

$$\sum_{L\in T_{\binom{n}{k}}} \Big(\prod_{\ell\in L}(\lambda_j - \lambda_\ell)\Big)\, |v_{L,M}|^2 = \big(\Lambda^k(\lambda_j I_n - A)\big)_{M,M} = \prod_{i=1}^{k}(\lambda_j - \mu_i^M)$$

where $\mu_1^M, \mu_2^M, \dots, \mu_k^M$ are the eigenvalues of $A_{M,M}$.

Proof. By the spectral theorem, $A = UDU^{-1}$, where $U \in GL_n(\mathbb{C})$ satisfies $U^* = U^{-1}$ and $D = diag[\lambda_1, \dots, \lambda_n]$. For $\tau \in \mathbb{C}$, let

$$D_\tau = diag[\tau - \lambda_1, \tau - \lambda_2, \dots, \tau - \lambda_n].$$

If $\tau \in \mathbb{C}$ then $\tau I_n - A = UD_\tau U^{-1}$ and

$$\Lambda^k(\tau I_n - A) = \Lambda^k(U)\Lambda^k(D_\tau)\Lambda^k(U)^{-1}.$$

We note that the columns of $\Lambda^k(U)$ are the wedge products v_L for $L \in T_{\binom{n}{k}}$. We compute directly that if $M, N \in T_{\binom{n}{k}}$ then

$$
\begin{aligned}
\left(\Lambda^k(\tau I_n - A)\right)_{M,N} &= \left(\Lambda^k(U)\Lambda^k(D_\tau)\Lambda^k(U)^{-1}\right)_{M,N} \\
&= \sum_{L,P\in T_{\binom{n}{k}}} (\Lambda^k(U))_{M,P}(\Lambda^k(D_\tau))_{P,L}(\Lambda^k(U^{-1}))_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Lambda^k(U))_{M,L}(\Lambda^k(D_\tau))_{L,L}(\Lambda^k(U^{-1}))_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Lambda^k(U))_{M,L}(\Lambda^k(D_\tau))_{L,L}(\Lambda^k(U)^*)_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Lambda^k(U))_{M,L}(\Lambda^k(D_\tau))_{L,L}\overline{(\Lambda^k(U))_{N,L}} \\
&= \sum_{L\in T_{\binom{n}{k}}} v_{L,M} \left(\prod_{\ell\in L}(\tau-\lambda_\ell)\right) \overline{v_{L,N}}.
\end{aligned}
$$

If $M = N$ and $\tau = \lambda_j$ for some $j \in \{1, 2, \ldots, n\}$ then,

$$
\begin{aligned}
\sum_{L\in T_{\binom{n}{k}}} \left(\prod_{\ell\in L}(\lambda_j-\lambda_\ell)\right) |v_{L,M}|^2 &= \sum_{L\in T_{\binom{n}{k}}} \left(\prod_{\ell\in L}(\lambda_j-\lambda_\ell)\right) v_{L,M}\, \overline{v_{L,M}} \\
&= \left(\Lambda^k(\lambda_j I_n - A)\right)_{M,M} \\
&= \prod_{i=1}^{k}(\lambda_j - \mu_i^M).
\end{aligned}
$$

□

Note that in Theorem 1.3.1, we did not assume that the eigenvalues of A are distinct. Equation (1.16) has a similar generalisation.

Theorem 1.3.2. *Let $n, k \in \mathbb{Z}_{>0}$ such that $k \leq n$. Let $A \in M_{n\times n}(\mathbb{C})$ satisfy $AA^* = A^*A$ and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Let $v_1, v_2, \ldots, v_n$ be the corresponding eigenvectors. If $L = \{l_1, l_2, \ldots, l_k\} \in T_{\binom{n}{k}}$ then define*

$$v_L = v_{l_1} \wedge v_{l_2} \wedge \cdots \wedge v_{l_k}.$$

For $M \in T_{\binom{n}{k}}$, let $v_{L,M} \in \mathbb{C}$ be the M component of v_L. If $\tau \in \mathbb{C}$ and $M, N \in T_{\binom{n}{k}}$ then

$$\sum_{L\in T_{\binom{n}{k}}} \Big(\prod_{\ell\in L^c}(\tau-\lambda_\ell)\Big)\, v_{L,M}\,\overline{v_{L,N}} = \big(\Upsilon^{n-k}(\tau I_n - A)\big)_{M,N}.$$

In particular, if $j\in\{1,2,\ldots,n\}$, $M=N=\{m_1,m_2,\ldots,m_k\}$ *and* A_{M^c,M^c} *is the* $(n-k)\times(n-k)$ *matrix formed by deleting rows* $m_1,m_2,\ldots,m_k$ *and columns* $m_1,m_2,\ldots,m_k$ *from* A *then*

$$\sum_{L\in T_{\binom{n}{k}}} \Big(\prod_{\ell\in L^c}(\lambda_j-\lambda_\ell)\Big)\, |v_{L,M}|^2 = \big(\Upsilon^{n-k}(\lambda_j I_n - A)\big)_{M,M} = \prod_{i=1}^{n-k}(\lambda_j-\mu_i^{M^c})$$

where $\mu_1^{M^c},\mu_2^{M^c},\ldots,\mu_{n-k}^{M^c}$ *are the eigenvalues of* A_{M^c,M^c}. *The complements* L^c *and* M^c *are taken with respect to the set* $\{1,2,\ldots,n\}$.

Proof. By the spectral theorem, $A=UDU^{-1}$, where $U\in GL_n(\mathbb{C})$ satisfies $U^*=U^{-1}$ and $D=diag[\lambda_1,\ldots,\lambda_n]$. For $\tau\in\mathbb{C}$, let

$$D_\tau = diag[\tau-\lambda_1,\tau-\lambda_2,\ldots,\tau-\lambda_n].$$

If $\tau\in\mathbb{C}$ then $\tau I_n - A = UD_\tau U^{-1}$ and

$$\Upsilon^{n-k}(\tau I_n - A) = \Upsilon^{n-k}(U)^{-1}\Upsilon^{n-k}(D_\tau)\Upsilon^{n-k}(U).$$

Since $U\in GL_n(\mathbb{C})$,

$$\Lambda^k(U)\Upsilon^{n-k}(U) = \Upsilon^{n-k}(U)\Lambda^k(U) = \det(U) I_{\binom{n}{k}}.$$

So,

$$\Upsilon^{n-k}(U) = \det(U)\Lambda^k(U)^{-1} \qquad \text{and} \qquad \Upsilon^{n-k}(U)^{-1} = \det(U)^{-1}\Lambda^k(U).$$

Moreover,

$$\big(\Upsilon^{n-k}(D_\tau)\big)_{M,M} = \big(\Lambda^{n-k}(D_\tau)\big)_{M^c,M^c}$$

Consequently, we compute for $M,N\in T_{\binom{n}{k}}$ that

$$
\begin{aligned}
\left(\Upsilon^{n-k}(\tau I_n - A)\right)_{M,N} &= \left(\Upsilon^{n-k}(U)^{-1}\Upsilon^{n-k}(D_\tau)\Upsilon^{n-k}(U)\right)_{M,N} \\
&= \sum_{L,P\in T_{\binom{n}{k}}} (\Upsilon^{n-k}(U)^{-1})_{M,P}(\Upsilon^{n-k}(D_\tau))_{P,L}(\Upsilon^{n-k}(U))_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Upsilon^{n-k}(U)^{-1})_{M,L}(\Upsilon^{n-k}(D_\tau))_{L,L}(\Upsilon^{n-k}(U))_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Lambda^{k}(U))_{M,L}(\Lambda^{n-k}(D_\tau))_{L^c,L^c}(\Lambda^{k}(U)^{-1})_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Lambda^{k}(U))_{M,L}(\Lambda^{n-k}(D_\tau))_{L^c,L^c}(\Lambda^{k}(U)^{*})_{L,N} \\
&= \sum_{L\in T_{\binom{n}{k}}} (\Lambda^{k}(U))_{M,L}(\Lambda^{n-k}(D_\tau))_{L^c,L^c}\overline{(\Lambda^{k}(U))_{N,L}} \\
&= \sum_{L\in T_{\binom{n}{k}}} v_{L,M}\left(\prod_{\ell\in L^c}(\tau-\lambda_\ell)\right)\overline{v_{L,N}}.
\end{aligned}
$$

Finally, we note that if $M = N$ and $\tau = \lambda_j$ for some $j \in \{1, 2, \ldots, n\}$ then,

$$
\begin{aligned}
\sum_{L\in T_{\binom{n}{k}}}\left(\prod_{\ell\in L^c}(\lambda_j-\lambda_\ell)\right)|v_{L,M}|^2 &= \sum_{L\in T_{\binom{n}{k}}}\left(\prod_{\ell\in L^c}(\lambda_j-\lambda_\ell)\right)v_{L,M}\,\overline{v_{L,M}} \\
&= \left(\Upsilon^{n-k}(\lambda_j I_n - A)\right)_{M,M} \\
&= \left(\Lambda^{n-k}(\lambda_j I_n - A)\right)_{M^c,M^c} \\
&= \prod_{i=1}^{n-k}(\lambda_j - \mu_i^{M^c}).
\end{aligned}
$$

□

We observe that in the statements of Theorem 1.3.1 and Theorem 1.3.2, we did not assume that A was Hermitian or that A had distinct eigenvalues. This will be reflected in the example which follows.

Example 1.3.1. Let

$$
A = \begin{pmatrix} 1+\frac{2}{3}i & -\frac{1}{3\sqrt{2}}-\frac{1}{3\sqrt{2}}i & -\frac{1}{3\sqrt{2}}+\frac{1}{3\sqrt{2}}i \\ \frac{1}{3\sqrt{2}}-\frac{1}{3\sqrt{2}}i & 1+\frac{1}{6}i & \frac{1}{6} \\ \frac{1}{3\sqrt{2}}+\frac{1}{3\sqrt{2}}i & -\frac{1}{6} & 1+\frac{1}{6}i \end{pmatrix}.
$$

Then, $A = UDU^*$, where $D = diag[1+i, 1, 1]$ and

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}i & \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}i & 0 \\ -\frac{1}{\sqrt{6}}i & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}i \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}i & \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}i \end{pmatrix}$$

is unitary. Let $\lambda_1 = 1+i$ and $\lambda_2 = \lambda_3 = 1$. In Theorem 1.3.2, set $n = 3$ and $k = 1$. Let $M = \{3\}$ so that $M^c = \{1, 2\}$. We compute directly that

$$\sum_{i=1}^{3} \big(\prod_{\ell \neq i}(\lambda_1 - \lambda_\ell)\big)\, |v_{i,3}|^2 = -\frac{1}{6}.$$

The eigenvalues of A_{M^c,M^c} are $\mu_1^{M^c} = 1$ and $\mu_2^{M^c} = 1 + \frac{5}{6}i$. Notice that

$$(\lambda_1 - \mu_1^{M^c})(\lambda_1 - \mu_2^{M^c}) = -\frac{1}{6}.$$

So,

$$\sum_{i=1}^{3} \big(\prod_{\ell \neq i}(\lambda_1 - \lambda_\ell)\big)\, |v_{i,3}|^2 = (\lambda_1 - \mu_1^{M^c})(\lambda_1 - \mu_2^{M^c})$$

which agrees with Theorem 1.3.2. Moreover, we can also compute that

$$\sum_{i=1}^{3} \big(\prod_{\ell \neq i}(0 - \lambda_\ell)\big)\, |v_{i,3}|^2 = 1 + \frac{5}{6}i = (0 - \mu_1^{M^c})(0 - \mu_2^{M^c})$$

which again agrees with Theorem 1.3.2.

Next, we will provide a second interpretation of Theorem 1.3.1 and Theorem 1.3.2. We know from Theorem 1.3.1 that

$$\sum_{L \in T_{\binom{n}{k}}} \Big(\prod_{\ell \in L}(\tau - \lambda_\ell)\Big)\, v_{L,M}\, \overline{v_{L,N}} = \big(\Lambda^k(\tau I_n - A)\big)_{M,N}.$$

Setting $M = N$ in $T_{\binom{n}{k}}$, we have

$$\sum_{L \in T_{\binom{n}{k}}} \Big(\prod_{\ell \in L}(\tau - \lambda_\ell)\Big)\, |v_{L,M}|^2 = \big(\Lambda^k(\tau I_n - A)\big)_{M,M}$$

By definition of Λ^k,

$$\big(\Lambda^k(\tau I_n - A)\big)_{M,M} = \det((\tau I_n - A)_{M,M}) = \det(\tau I_k - A_{M,M}).$$

But, $\det(\tau I_k - A_{M,M})$ is the characteristic polynomial of the $k \times k$ matrix $A_{M,M}$. Consequently, we have the corollary

Corollary 1.3.3. *Let $n, k \in \mathbb{Z}_{>0}$ such that $k \leq n$. Let $A \in M_{n\times n}(\mathbb{C})$ satisfy $AA^* = A^*A$ and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. If $M \in T_{\binom{n}{k}}$ then*

$$p_M(\tau) = \sum_{L\in T_{\binom{n}{k}}} |v_{L,M}|^2 \Big(\prod_{\ell\in L} (\tau - \lambda_\ell) \Big)$$

is the characteristic polynomial of the $k \times k$ matrix $A_{M,M}$. Moreover,

$$q_M(\tau) = \sum_{L\in T_{\binom{n}{k}}} |v_{L,M}|^2 \Big(\prod_{\ell\in L^c} (\tau - \lambda_\ell) \Big)$$

is the characteristic polynomial of the $(n-k) \times (n-k)$ matrix A_{M^c,M^c}.

1.4 Eigenvectors of minor matrices — an algorithm

Let $A \in M_{n\times n}(\mathbb{C})$ be a matrix satisfying $AA^* = A^*A$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. By the spectral theorem, there exists a unitary matrix $U \in GL_n(\mathbb{C})$ such that

$$A = U diag[\lambda_1, \ldots, \lambda_n] U^*.$$

Let $M \in T_{\binom{n}{k}}$, where $k \in \{1, 2, \ldots, n-1\}$. We have the following theorem, which applies to A if the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct.

Theorem 1.4.1. *Let $A \in M_{n\times n}(\mathbb{C})$ and $\lambda_1, \ldots, \lambda_n$ be distinct eigenvalues of A with corresponding eigenvectors $v_1, \ldots, v_n$. Suppose that there exists a unitary matrix U such that $A = U diag[\lambda_1, \ldots, \lambda_n]U^*$. If $L = \{i_1, \ldots, i_k\} \in T_{\binom{n}{k}}$ with $k \in \{1, \ldots, n-1\}$, define $v_L = v_{i_1} \wedge \cdots \wedge v_{i_k}$. If $M, P \in T_{\binom{n}{k}}$ and $v_{L,M}$ is the M element of v_L then*

$$v_{L,M}\overline{v_{L,P}} = \frac{1}{\Big(\prod_{l\in L,\ a\in L^c} (\lambda_a - \lambda_l)\Big)} \big(\Lambda^k (\prod_{a\in L^c} (\lambda_a I_n - A))\big)_{M,P}. \tag{1.17}$$

We will provide an algorithm for computing the eigenvectors of the $k \times k$ matrix $A_{M,M}$ in the special case where the eigenvalues of $A_{M,M}$ are **distinct**. Our algorithm is based heavily on Corollary 1.3.3 and Theorem 1.4.1.

1. We will denote the i, j element of A by a_{ij}. By Corollary 1.3.3, the characteristic polynomial of $A_{M,M}$ is

$$p_M(\tau) = \sum_{L \in T_{\binom{n}{k}}} |v_{L,M}|^2 \left(\prod_{\ell \in L} (\tau - \lambda_\ell) \right).$$

   One can compute the coefficients $|v_{L,M}|^2$ by diagonalising A and then computing them directly or by using equation (1.17) if the eigenvalues of A are distinct.

2. After computing the characteristic polynomial $p_M(\tau)$, find its k roots, which are the eigenvalues $\mu_1^M, \dots, \mu_k^M$ of $A_{M,M}$. Recall that $\mu_1^M, \dots, \mu_k^M$ are distinct by assumption.

3. The assumption that the eigenvalues of $A_{M,M}$ are distinct means that we can apply Theorem 1.4.1 to obtain for $l, p, q \in \{1, 2, \dots, k\}$

$$v_{l,p}\overline{v_{l,q}} = \frac{1}{\left(\prod_{a \neq l} (\mu_a^M - \mu_l^M) \right)} \left(\prod_{a \neq l} (\mu_a^M I_k - A) \right)_{p,q}.$$

   We obtain k^2 equations, which we can solve to obtain the elements $v_{l,p}$ and hence, the eigenvectors $[v_{l,1}, v_{l,2}, \dots, v_{l,k}]^T$ of $A_{M,M}$ for $l \in \{1, 2, \dots, k\}$.

Let us give concrete examples of the algorithm in action.

Example 1.4.1. Let

$$A = \begin{pmatrix} 1 + \frac{2}{3}i & -\frac{1}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}i & -\frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}}i \\ \frac{1}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}i & 1 + \frac{1}{6}i & \frac{1}{6} \\ \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}}i & -\frac{1}{6} & 1 + \frac{1}{6}i \end{pmatrix}.$$

Recall that the eigenvalues of A are $\lambda_1 = 1 + i$ and $\lambda_2 = \lambda_3 = 1$. We will use the algorithm to compute the eigenvectors of the 2×2 matrix

$$A_{\{1,2\},\{1,2\}} = \begin{pmatrix} 1 + \frac{2}{3}i & -\frac{1}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}i \\ \frac{1}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}i & 1 + \frac{1}{6}i \end{pmatrix}.$$

Step 1: By Corollary 1.3.3, we need to compute the coefficients

$$|v_{\{1,2\},\{1,2\}}|^2, \qquad |v_{\{1,2\},\{1,3\}}|^2 \qquad \text{and} \qquad |v_{\{1,2\},\{2,3\}}|^2.$$

Since A does not have distinct eigenvalues, we proceed by direct computation. We find that $|v_{\{1,2\},\{1,2\}}|^2 = 1/2$, $|v_{\{1,2\},\{1,3\}}|^2 = 1/3$ and $|v_{\{1,2\},\{2,3\}}|^2 = 1/6$. So, by Corollary 1.3.3,

$$\begin{aligned} p_{\{1,2\}}(\tau) &= \frac{1}{2}(\tau - 1 - i)(\tau - 1) + \frac{1}{3}(\tau - 1 - i)(\tau - 1) + \frac{1}{6}(\tau - 1)^2 \\ &= \tau^2 - (2 + \frac{5}{6}i)\tau + (1 + \frac{5}{6}i) \\ &= (\tau - 1)(\tau - 1 - \frac{5}{6}i). \end{aligned}$$

Step 2: The eigenvalues of $A_{\{1,2\},\{1,2\}}$ are therefore, $\mu_1 = 1$ and $\mu_2 = 1 + \frac{5}{6}i$. This is obtained by finding the roots of the characteristic polynomial $p_{\{1,2\}}(\tau) = (\tau - 1)(\tau - 1 - \frac{5}{6}i)$.

Step 3: Let w_1, w_2 be the eigenvectors corresponding to μ_1 and μ_2. Since $A_{\{1,2\},\{1,2\}}$ has distinct eigenvalues, we can apply Theorem 1.4.1 to find that

1. $|w_{1,1}|^2 = \frac{1}{\mu_2 - \mu_1}(\mu_2 I_2 - A_{\{1,2\},\{1,2\}})_{1,1} = 1/5$.
2. $|w_{1,2}|^2 = \frac{1}{\mu_2 - \mu_1}(\mu_2 I_2 - A_{\{1,2\},\{1,2\}})_{2,2} = 4/5$.
3. $w_{1,1}\overline{w_{1,2}} = \frac{1}{\mu_2 - \mu_1}(\mu_2 I_2 - A_{\{1,2\},\{1,2\}})_{1,2} = \frac{\sqrt{2}}{5} - \frac{\sqrt{2}}{5}i$.
4. $\overline{w_{1,1}}w_{1,2} = \frac{1}{\mu_2 - \mu_1}(\mu_2 I_2 - A_{\{1,2\},\{1,2\}})_{2,1} = \frac{\sqrt{2}}{5} + \frac{\sqrt{2}}{5}i$.
5. $|w_{2,1}|^2 = \frac{1}{\mu_1 - \mu_2}(\mu_1 I_2 - A_{\{1,2\},\{1,2\}})_{1,1} = 4/5$.
6. $|w_{2,2}|^2 = \frac{1}{\mu_1 - \mu_2}(\mu_1 I_2 - A_{\{1,2\},\{1,2\}})_{2,2} = 1/5$.
7. $w_{2,1}\overline{w_{2,2}} = \frac{1}{\mu_1 - \mu_2}(\mu_1 I_2 - A_{\{1,2\},\{1,2\}})_{1,2} = -\frac{\sqrt{2}}{5} + \frac{\sqrt{2}}{5}i$.
8. $\overline{w_{2,1}}w_{2,2} = \frac{1}{\mu_1 - \mu_2}(\mu_1 I_2 - A_{\{1,2\},\{1,2\}})_{2,1} = -\frac{\sqrt{2}}{5} - \frac{\sqrt{2}}{5}i$.

From the computations, we can write

1. $w_{1,1} = |w_{1,1}|e^{i\theta_1} = \frac{1}{\sqrt{5}}e^{i\theta_1}$
2. $w_{1,2} = |w_{1,2}|e^{i\theta_2} = \frac{2}{\sqrt{5}}e^{i\theta_2}$
3. $w_{2,1} = |w_{2,1}|e^{i\alpha_1} = \frac{2}{\sqrt{5}}e^{i\alpha_1}$
4. $w_{2,2} = |w_{2,2}|e^{i\alpha_2} = \frac{1}{\sqrt{5}}e^{i\alpha_2}$

where $\theta_1, \theta_2, \alpha_1, \alpha_2 \in (-\pi, \pi]$. Upon substitution into the equations for $w_{1,1}\overline{w_{1,2}}$ and $w_{2,1}\overline{w_{2,2}}$, we deduce that

$$e^{i(\theta_1-\theta_2)} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = e^{-i\frac{\pi}{4}}$$

and

$$e^{i(\alpha_1-\alpha_2)} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = e^{i\frac{3\pi}{4}}.$$

So, $\theta_1 - \theta_2 = -\pi/4$ and $\alpha_1 - \alpha_2 = 3\pi/4$. We can set $\theta_1 = \alpha_1 = 0$ so that $\theta_2 = \pi/4$ and $\alpha_2 = -3\pi/4$. Hence,

$$w_{1,1} = \frac{1}{\sqrt{5}}, \quad w_{1,2} = \frac{2}{\sqrt{5}}e^{\pi i/4}, \quad w_{2,1} = \frac{2}{\sqrt{5}} \quad \text{and} \quad w_{2,2} = \frac{1}{\sqrt{5}}e^{-3\pi i/4}.$$

One can check that the matrix

$$W = \begin{pmatrix} w_{1,1} & w_{2,1} \\ w_{1,2} & w_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}}e^{\pi i/4} & \frac{1}{\sqrt{5}}e^{-3\pi i/4} \end{pmatrix}$$

satisfies $W^{-1}A_{\{1,2\},\{1,2\}}W = diag[1, 1 + \frac{5}{6}i]$ as required.

Note that we have freedom in choosing the angles θ_1 and α_1 because $\theta_2 = \theta_1 + \pi/4$ and $\alpha_2 = \alpha_1 - 3\pi/4$. For instance, if we choose $\theta_1 = \pi/4$ and $\alpha_1 = \pi/2$ then

$$W = \begin{pmatrix} \frac{1}{\sqrt{5}}e^{\pi i/4} & \frac{2}{\sqrt{5}}e^{i\pi/2} \\ \frac{2}{\sqrt{5}}e^{\pi i/2} & \frac{1}{\sqrt{5}}e^{-\pi i/4} \end{pmatrix}$$

which still satisfies the equation $W^{-1}A_{\{1,2\},\{1,2\}}W = diag[1, 1 + \frac{5}{6}i]$.

Bibliography

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