# Examples of six-term exact sequences in K-theory

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## Contents



## 1 A quick summary of the six-term exact sequence

Most of the material in this document originates from the excellent reference [RLL00]. The purpose of K-theory is to distinguish C*-algebras from each other. K-theory has seen success in the classification of AF-algebras, an important result due to Elliott ([RLL00, Section 7]), and the Kirchberg-Phillips classification of Kirchberg algebras satisfying the UCT in [Phi00].

Given a C*-algebra $A$, it is often useful to compute the groups $K_0(A)$ and $K_1(A)$. If $A$ lies in a short exact sequence of C*-algebras then one can use the *six-term exact sequence* (see [RLL00, Theorem 12.1.2]) to compute $K_0(A)$ and $K_1(A)$. To be specific, let

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

be a short exact sequence of C*-algebras. This induces the six-term exact sequence

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\varphi)} & K_1(A) & \xrightarrow{K_1(\psi)} & K_1(B) \\ \delta_0\uparrow & & & & \downarrow\delta_1 \\ K_0(B) & \xleftarrow[K_0(\psi)]{} & K_0(A) & \xleftarrow[K_0(\varphi)]{} & K_0(I) \end{array}$$

The group homomorphisms $\delta_1$ and $\delta_0$ are called the index map and exponential map respectively. We now briefly recall their definitions from [RLL00]. In what follows, we will freely use notation from [RLL00].

The index map is constructed from [RLL00, Lemma 9.11, Lemma 9.12].

**Lemma 1.1.** *Let*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

*be a short exact sequence of $C^*$-algebras. Let $u \in \mathcal{U}_n(\tilde{B})$. Then*

1. *There exist $v \in \mathcal{U}_{2n}(\tilde{A})$ and $p \in \mathcal{P}_{2n}(\tilde{I})$ such that*

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad \tilde{\varphi}(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \qquad s(p) = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}.$$

2. *If $w \in \mathcal{U}_{2n}(\tilde{A})$ and $q \in \mathcal{P}_{2n}(\tilde{I})$ satisfy*

$$\tilde{\psi}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \qquad \text{and} \qquad \tilde{\varphi}(q) = w \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} w^*.$$

   *then $s(q) = diag[1_n, 0_n]$ and $p \sim_u q$ in $\mathcal{P}_{2n}(\tilde{I})$.*

In Lemma 1.1, $s : M_n(\tilde{I}) \to M_n(\tilde{I})$ is the *scalar mapping*, defined in [RLL00, Section 4.2.1]. From Lemma 1.1, define the map

$$\begin{array}{rccc} \nu: & \mathcal{U}_\infty(\tilde{B}) & \to & K_0(I) \\ & u & \mapsto & [p]_0 - [s(p)]_0 \end{array}$$

where $p \in \mathcal{P}_{2n}(\tilde{I})$ is the element constructed in Lemma 1.1 from $u$. By the standard picture for $K_0$ ([RLL00, Proposition 4.2.2]),

$$K_0(I) = \{[q]_0 - [s(q)]_0 \mid q \in \mathcal{P}_\infty(\tilde{I})\}.$$

So $\nu$ is a well-defined map into $K_0(I)$. By [RLL00, Lemma 9.1.2], this map satisfies the properties needed for us to apply the universal property of $K_1$ in [RLL00, Proposition 8.1.5]. Hence, we obtain the unique group morphism

$$\begin{array}{rccc}\delta_1: & K_1(B) & \to & K_0(I)\\ & [u]_1 & \mapsto & \nu(u).\end{array}$$

The exponential map $\delta_0$ is defined by using the index map. In particular, it is the composite

$$K_0(B) \xrightarrow{\beta} K_2(B) \xrightarrow{\delta_2} K_1(I)$$

where $\beta : K_0(B) \to K_2(B)$ is the Bott map which is well-known to be an isomorphism (Bott periodicity, see [RLL00, Section 11.2]) and $\delta_2$ is a higher index map; the unique group morphism making the following diagram commute:

$$\begin{array}{ccc} K_2(B) & \overset{\delta_2}{\dashrightarrow} & K_1(I) \\ \downarrow\cong & & \downarrow\cong \\ K_1(SB) & \xrightarrow[\delta_{1,S}]{} & K_0(SI) \end{array}$$

The map $\delta_{1,S}$ is the index map associated to the short exact sequence of suspensions

$$0 \longrightarrow SI \longrightarrow SA \longrightarrow SB \longrightarrow 0.$$

The most important thing about the exponential map is that there is a specific way to compute it, as highlighted by [RLL00, Proposition 12.2.2].

**Theorem 1.2.** *Let*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

*be a short exact sequence of $C^*$-algebras and $\delta_0 : K_0(B) \to K_1(I)$ be the associated exponential map. Let $g \in K_0(B)$ and $p \in \mathcal{P}_n(\tilde{B})$ be such that $g = [p]_0 - [s(p)]_0$ (standard picture of $K_0$). Let $a \in M_n(\tilde{A})$ be self-adjoint and satisfy $\tilde{\psi}(a) = p$. Then there exists a unique unitary $u \in \mathcal{U}_n(\tilde{I})$ such that*

$$\tilde{\varphi}(u) = \exp(2\pi i a) \qquad \textit{and} \qquad \delta_0(g) = -[u]_1.$$

Theorem 1.2 simplifies further when one considers a unital extension.

**Theorem 1.3.** *Let*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

*be a short exact sequence of C*-algebras where A is unital (this forces B to be unital and ψ to be unital preserving). Let* $\delta_0 : K_0(B) \to K_1(I)$ *be the associated exponential map. Let* $g \in K_0(B)$ *and* $p \in \mathcal{P}_n(\tilde{B})$ *be such that* $g = [p]_0$ *(standard picture of* $K_0$*). Let* $a \in M_n(\tilde{A})$ *be self-adjoint and satisfy* $\psi(a) = p$*. Define the *-homomorphism*

$$\begin{array}{cccc} \overline{\varphi} : & \tilde{I} & \to & A \\ & x + \alpha 1_{\tilde{I}} & \mapsto & \varphi(x) + \alpha 1_A. \end{array}$$

*Then there exists a unique unitary* $u \in \mathcal{U}_n(\tilde{I})$ *such that*

$$\overline{\varphi}(u) = \exp(2\pi i a) \qquad \text{and} \qquad \delta_0(g) = -[u]_1.$$

Theorem 1.3 can be broken down into the following steps:

1. Let $n \in \mathbb{Z}_{>0}$ and $p \in \mathcal{P}_n(\tilde{B})$ so that $[p]_0 \in K_0(B)$.

2. There exists a self-adjoint element $a \in M_n(\tilde{A})$ such that $\psi(a) = p$. Note that $\psi$ is extended to $M_n(\tilde{A})$ in the obvious manner.

3. Take the exponential to obtain $u = \exp(2\pi i a) \in \mathcal{U}_n(A)$.

4. Observe that by the continuous functional calculus,

$$\psi(u) = \exp(2\pi i \psi(a)) = \exp(2\pi i p) = 1_B.$$

   The last equality follows from the fact that $p$ is a projection (and $\sigma(p) = \{0, 1\}$).

5. The calculation in the previous step demonstrates that $1_A - u \in \ker \psi = \operatorname{im} \varphi$. So there exists $v \in M_n(I)$ such that $\varphi(v) = 1_A - u$.

6. The element $1_{\tilde{I}} - v \in M_n(\tilde{I})$ satisfies

$$\overline{\varphi}(1_{\tilde{I}} - v) = 1_A - (1_A - u) = u.$$

   Notice that $1_{\tilde{I}} - v$ is unitary because $\overline{\varphi}$ is injective and

$$\overline{\varphi}((1_{\tilde{I}} - v)^*(1_{\tilde{I}} - v)) = u^* u = \overline{\varphi}(1_{\tilde{I}}).$$

7. By Theorem 1.3, $\delta_0([p]_0) = -[1_{\tilde{I}} - v]_1$.

# 2 Examples of six-term exact sequences

Most of the listed examples originate from [RLL00, Exercise 12.4]. In our computations, we will make use of the tables in [RLL00, Pages 234-235], which depict the $K_0$ and $K_1$ groups of well-known C*-algebras.

**Example 2.1.** Let $A$ be a non-unital C*-algebra. Then we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

To be clear, $\tilde{A}$ is the unitisation of $A$. This induces the six-term exact sequence

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{K_1(\iota)} & K_1(\tilde{A}) & \xrightarrow{K_1(\pi)} & K_1(\mathbb{C}) \\ \delta_0 \big\uparrow & & & & \big\downarrow \delta_1 \\ K_0(\mathbb{C}) & \xleftarrow[K_0(\pi)]{} & K_0(\tilde{A}) & \xleftarrow[K_0(\iota)]{} & K_0(A) \end{array}$$

Recall that $K_0(\mathbb{C}) \cong \mathbb{Z}$ (rank!) and $K_1(\mathbb{C}) = 0$. So our six-term exact sequence becomes

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{K_1(\iota)} & K_1(\tilde{A}) & \xrightarrow{K_1(\pi)} & 0 \\ \delta_0 \big\uparrow & & & & \big\downarrow \delta_1 \\ \mathbb{Z} & \xleftarrow[K_0(\pi)]{} & K_0(\tilde{A}) & \xleftarrow[K_0(\iota)]{} & K_0(A) \end{array}$$

We see that the index map $\delta_1$ is the zero map. Now recall by definition that $K_1(A) = K_1(\tilde{A})$. This means that $K_1(\iota)$ is a group isomorphism. By exactness,

$$\text{im } \delta_0 = \ker K_1(\iota) = \{0\}.$$

Hence, the exponential map $\delta_0$ must also be the zero map. By using exactness again, $K_0(\pi)$ is surjective, $K_0(\iota)$ is injective and

$$K_0(A) \cong \text{im } K_0(\iota) = \ker K_0(\pi).$$

This is consistent with how $K_0(A)$ is defined when $A$ is non-unital. The definitions of $K_0(A)$ and $K_1(A)$ when $A$ is non-unital are designed specifically to make the associated six-term sequence exact.

**Example 2.2.** Let $H$ be an infinite-dimensional separable Hilbert space. We have the short exact sequence

$$0 \longrightarrow K \xrightarrow{\iota} B(H) \xrightarrow{\pi} Q(H) \longrightarrow 0.$$

where $Q(H)$ is the Calkin algebra. This induces the six-term exact sequence

$$\begin{array}{ccccc} K_1(K) & \xrightarrow{K_1(\iota)} & K_1(B(H)) & \xrightarrow{K_1(\pi)} & K_1(Q(H)) \\ {\scriptstyle \delta_0}\uparrow & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(Q(H)) & \xleftarrow[K_0(\pi)]{} & K_0(B(H)) & \xleftarrow[K_0(\iota)]{} & K_0(K) \end{array}$$

Now recall that $K_0(K) \cong \mathbb{Z}$, $K_1(K) = 0$, $K_0(B(H)) = 0$ and $K_1(B(H)) = 0$. So the six-term exact sequence drastically reduces to

$$\begin{array}{ccccc} 0 & \xrightarrow{K_1(\iota)} & 0 & \xrightarrow{K_1(\pi)} & K_1(Q(H)) \\ {\scriptstyle \delta_0}\uparrow & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(Q(H)) & \xleftarrow[K_0(\pi)]{} & 0 & \xleftarrow[K_0(\iota)]{} & \mathbb{Z} \end{array}$$

In particular, we see by exactness that the index map $\delta_1$ is an isomorphism. So $K_1(Q(H)) \cong \mathbb{Z}$. Similarly, the exponential map $\delta_0$ is also an isomorphism. So $K_0(Q(H)) = 0$.

**Example 2.3.** Let $A$ be a C*-algebra. Recall the suspension and cone of $A$ are defined by

$$SA = C_0((0,1)) \otimes A \qquad \text{and} \qquad CA = C_0((0,1]) \otimes A.$$

respectively. To be more explicit,

$$SA = \{f \in C([0,1], A) \mid f(0) = f(1) = 0\}$$

and

$$CA = \{f \in C([0,1], A) \mid f(0) = 0\}.$$

We have a short exact sequence

$$0 \longrightarrow SA \xrightarrow{\iota} CA \xrightarrow{f} A \longrightarrow 0.$$

where if $\phi \in CA$ then $f(\phi) = \phi(1)$. This yields the six-term exact sequence

$$\begin{array}{ccccc}
K_1(SA) & \xrightarrow{K_1(\iota)} & K_1(CA) & \xrightarrow{K_1(f)} & K_1(A) \\
{\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\
K_0(A) & \xleftarrow[K_0(f)]{} & K_0(CA) & \xleftarrow[K_0(\iota)]{} & K_0(SA)
\end{array}$$

Recall that the cone $CA$ is homotopic to zero. Therefore $K_0(CA) = K_1(CA) = 0$. Our six-term exact sequence becomes

$$\begin{array}{ccccc}
K_1(SA) & \xrightarrow{K_1(\iota)} & 0 & \xrightarrow{K_1(f)} & K_1(A) \\
{\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\
K_0(A) & \xleftarrow[K_0(f)]{} & 0 & \xleftarrow[K_0(\iota)]{} & K_0(SA)
\end{array}$$

and by exactness, both the index and exponential maps are isomorphisms.

**Example 2.4.** We have the short exact sequence

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{f} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

where if $\phi \in C([0,1])$ then $f(\phi) = (\phi(0), \phi(1))$. This induces the six-term exact sequence

$$\begin{array}{ccccc}
K_1(C_0((0,1))) & \xrightarrow{K_1(\iota)} & K_1(C([0,1])) & \xrightarrow{K_1(f)} & K_1(\mathbb{C} \oplus \mathbb{C}) \\
{\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\
K_0(\mathbb{C} \oplus \mathbb{C}) & \xleftarrow[K_0(f)]{} & K_0(C([0,1])) & \xleftarrow[K_0(\iota)]{} & K_0(C_0((0,1)))
\end{array}$$

Now $K_1(\mathbb{C} \oplus \mathbb{C}) = 0$ and $K_0(\mathbb{C} \oplus \mathbb{C}) = \mathbb{Z}^2$ (use the fact that $K_0$ preserves direct sums). Observe that $C_0((0,1)) = S\mathbb{C}$. So

$$K_0(C_0((0,1))) \cong K_1(\mathbb{C}) = 0 \qquad \text{and} \qquad K_1(C_0((0,1))) \cong K_0(\mathbb{C}) = \mathbb{Z}.$$

Our six-term exact sequence now becomes

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{K_1(\iota)} & K_1(C([0,1])) & \xrightarrow{K_1(f)} & 0 \\
{\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\
\mathbb{Z}^2 & \xleftarrow[K_0(f)]{} & K_0(C([0,1])) & \xleftarrow[K_0(\iota)]{} & 0
\end{array}$$

The index map $\delta_1$ is simply the zero map. Now the exponential map $\delta_0$ is a group homomorphism from $\mathbb{Z}^2$ to $\mathbb{Z}$. Let us compute it using Theorem 1.3.

The C*-algebra $\mathbb{C} \oplus \mathbb{C}$ is generated by the set $\{(1,0),(0,1)\}$. Let us compute $\delta_0([(1,0)]_0)$ and $\delta_0([(0,1)]_0)$.

Consider the self-adjoint element $1 - id_{[0,1]} \in C([0,1])$. It satisfies

$$f(1 - id_{[0,1]}) = (1-0, 1-1) = (1,0) \in \mathbb{C} \oplus \mathbb{C}.$$

By taking the exponential, define

$$u = \exp(2\pi i(1 - id_{[0,1]})) = \exp(-2\pi i \cdot id_{[0,1]}) \in \mathcal{U}(C[0,1]).$$

Then $1 - u \in \ker f = \operatorname{im} \iota$. To be explicit, $1-u$ is the function

$$\begin{array}{rccc} 1-u: & [0,1] & \to & \mathbb{C} \\ & t & \mapsto & 1-\exp(-2\pi i t). \end{array}$$

So $\iota(1-u) = 1-u$. Subsequently the element $1-(1-u) = u$ is a unitary element in the unitisation of $C_0((0,1))$ satisfying

$$\bar{\iota}(u) = 1 - \iota(1-u) = 1-(1-u) = u \in C([0,1]).$$

By Theorem 1.3,

$$\delta_0([(1,0)]_0) = -[u]_1 = -[\exp(-2\pi i \cdot id_{[0,1]})]_1.$$

Similarly,

$$\delta_0([(0,1)]_0) = -[\exp(2\pi i \cdot id_{[0,1]})]_1.$$

Now the image of $\delta_0$ lands in $K_1(C_0((0,1))) = K_1(S\mathbb{C})$. We have the Bott isomorphism $\beta : K_0(\mathbb{C}) \to K_1(S\mathbb{C})$ (see [RLL00, Theorem 11.1.2]). If $p \in \mathcal{P}_\infty(\mathbb{C})$ is a projection then $\beta([p]_0) = [f_p]_0$ where $f_p \in \mathcal{U}((S\mathbb{C})^\sim)$ is the projection loop defined for $t \in [0,1]$ by

$$\begin{aligned} f_p(t) &= \exp(2\pi i t)p + (1-p) = \sum_{j=0}^{\infty} \frac{(2\pi i t)^j p}{j!} + (1-p) \\ &= 1 + \sum_{j=1}^{\infty} \frac{(2\pi i t)^j p}{j!} = \sum_{j=0}^{\infty} \frac{(2\pi i t p)^j}{j!} \\ &= \exp(2\pi i t p). \end{aligned}$$

In particular, the group $K_0(\mathbb{C})$ is generated by $[1]_0$ and

$$\beta([1]_0) = [f_1]_1 = [t \mapsto \exp(2\pi i t(1))]_1 = [\exp(2\pi i \cdot id_{[0,1]})]_1.$$

This calculation shows that $[\exp(2\pi i \cdot id_{[0,1]})]_1$ generates $K_1(S\mathbb{C})$. We also have

$$\beta([-1]_0) = [f_{-1}]_1 = [t \mapsto \exp(2\pi i t(-1))]_1 = [\exp(-2\pi i \cdot id_{[0,1]})]_1.$$

and consequently, $\delta_0$ is the group homomorphism

$$\begin{array}{rccc} \delta_0 : & \mathbb{Z}^2 & \to & \mathbb{Z} \\ & (1,0) & \mapsto & 1 \\ & (0,1) & \mapsto & -1 \end{array}$$

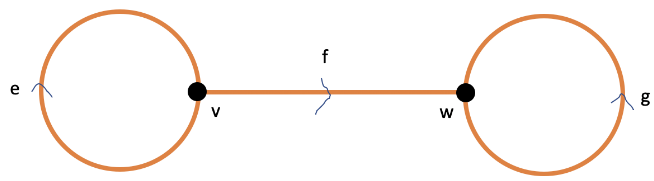
In particular, $\delta_0$ is surjective. By exactness, $K_1(\iota)$ is the zero map and $\ker K_1(f) = \operatorname{im} K_1(\iota) = \{0\}$. We deduce that the zero map $K_1(f)$ is actually a group isomorphism. So $K_1(C([0,1])) = 0$.

Now $K_0(f)$ is injective and its image is

$$\ker \delta_0 = \{(n,n) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Therefore $K_0(C([0,1])) \cong \mathbb{Z}$.

**Example 2.5.** Let $\mathcal{G}$ be the directed graph




and $C^*(\mathcal{G})$ be the associated graph C*-algebra (see [Rae05] for an introduction). To be specific, $C^*(\mathcal{G})$ is generated by the union of the set of partial isometries $\{S_e, S_f, S_g\}$ and the set of mutually orthogonal projections $\{P_v, P_w\}$. Moreover, these elements satisfy the Cuntz-Krieger relations (see [Rae05, Page 6])

$$P_v = S_e^* S_e = S_f^* S_f = S_e S_e^*, \qquad P_w = S_g^* S_g = S_f S_f^* + S_g S_g^*.$$

We have the short exact sequence

$$0 \longrightarrow C(S^1) \otimes K \xrightarrow{\iota} C^*(\mathcal{G}) \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

where $C(S^1)$ is isomorphic to the graph C*-algebra associated to the single loop at $w$ (see [Rae05, Theorem 4.9]). This induces the six-term short exact sequence

$$\begin{array}{ccccc} K_1(C(S^1)\otimes K) & \xrightarrow{K_1(\iota)} & K_1(C^*(\mathcal{G})) & \xrightarrow{K_1(\pi)} & K_1(C(S^1)) \\ {\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\ K_0(C(S^1)) & \xleftarrow[K_0(\pi)]{} & K_0(C^*(\mathcal{G})) & \xleftarrow[K_0(\iota)]{} & K_0(C(S^1)\otimes K) \end{array}$$

We know that $K_0(C(S^1)) \cong \mathbb{Z} \cong K_1(C(S^1))$, $K_0(C(S^1)) \cong K_0(C(S^1)\otimes K)$ and $K_1(C(S^1)) \cong K_1(C(S^1)\otimes K)$. Hence our six-term exact sequence becomes

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{K_1(\iota)} & K_1(C^*(\mathcal{G})) & \xrightarrow{K_1(\pi)} & \mathbb{Z} \\ {\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\ \mathbb{Z} & \xleftarrow[K_0(\pi)]{} & K_0(C^*(\mathcal{G})) & \xleftarrow[K_0(\iota)]{} & \mathbb{Z} \end{array}$$

First, observe that in $K_0(C^*(\mathcal{G}))$,

$$[P_w]_0 = [S_gS_g^* + S_fS_f^*]_0 = [S_g^*S_g]_0 + [S_f^*S_f]_0 = [P_w]_0 + [P_v]_0.$$

So $[P_v]_0 = 0$ in $K_0(C^*(\mathcal{G}))$ and $K_0(\iota)$ is the zero map. By exactness, im $\delta_1 = \ker K_0(\iota) = \mathbb{Z}$. So the index map $\delta_1$ is a surjective group morphism from $\mathbb{Z}$ to $\mathbb{Z}$. There are only two such group morphisms — the identity map $id_{\mathbb{Z}}$ and $-id_{\mathbb{Z}}$. In either case, we deduce that the index map $\delta_1$ is a group isomorphism.

Now by exactness, im $K_1(\pi) = \ker \delta_1 = \{0\}$. So, $K_1(\pi)$ is the zero map. Using exactness again, we find that $K_1(C^*(\mathcal{G})) = \ker K_1(\pi) =$ im $K_1(\iota)$. Therefore $K_1(\iota)$ is surjective.

Since $K_0(\iota)$ is the zero map then by exactness $K_0(\pi)$ is injective. Consequently, we obtain the exact sequence of groups

$$0 \longrightarrow K_0(C^*(\mathcal{G})) \xrightarrow{K_0(\pi)} \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z} \xrightarrow{K_1(\iota)} K_1(C^*(\mathcal{G})) \longrightarrow 0. \qquad (1)$$

We will now compute the exponential map $\delta_0$, a group morphism from $K_0(C(S^1))$ to $K_1(C(S^1)\otimes K)$. By identifying the quotient $C(S^1)$ with the graph C*-algebra associated to the single loop at vertex $w$, we find that as a group, $K_0(C(S^1))$ is generated by $[\pi(P_w)]_0$. Hence, it suffices to compute $\delta_0([\pi(P_w)]_0)$.

Obviously, $P_w \in C^*(\mathcal{G})$ maps to $\pi(P_w)$ via $\pi$. Notice however that $P_w$ is not just a self-adjoint element; it is also a projection. So the exponential

$$\exp(2\pi i P_w) = 1_{C^*(\mathcal{G})} = P_v + P_w.$$

Now set $I = C(S^1) \otimes K$ so that the unitisation is denoted by $\tilde{I}$. The unit $1_{\tilde{I}}$ is a unitary element satisfying $\bar{\iota}(1_{\tilde{I}}) = 1_{C^*(\mathcal{G})}$. By Theorem 1.3,

$$\delta_0([\pi(P_w)]_0) = -[1_{\tilde{I}}]_0.$$

Since $[\pi(P_w)]_0$ generates $K_0(C(S^1))$ then the exponential map $\delta_0$ must be the zero map.

Now consider the exact sequence in equation (1) again. By exactness, we find that im $K_0(\pi) = \ker \delta_0 = \mathbb{Z}$ and $\ker K_1(\iota) = \text{im } \delta_0 = \{0\}$. Therefore $K_0(\pi)$ and $K_1(\iota)$ are both group isomorphisms. So

$$K_0(C^*(\mathcal{G})) \cong \mathbb{Z} \qquad \text{and} \qquad K_1(C^*(\mathcal{G})) \cong \mathbb{Z}.$$

**Example 2.6.** This example is taken from [RLL00, Section 13.1]. If $n \in \mathbb{Z}_{>1}$ then define the dimension drop algebra by

$$D_n = \{f \in C([0,1], M_n(\mathbb{C})) \mid f(0) = 0,\ f(1) \in \mathbb{C}1_n\}.$$

We have a short exact sequence

$$0 \longrightarrow SM_n(\mathbb{C}) \xrightarrow{\iota} D_n \xrightarrow{f} \mathbb{C} \longrightarrow 0.$$

where if $\phi \in D_n$ then $f(\phi) = \phi(1)$. We now have the six-term exact sequence

$$\begin{array}{ccccc} K_1(SM_n(\mathbb{C})) & \xrightarrow{K_1(\iota)} & K_1(D_n) & \xrightarrow{K_1(f)} & K_1(\mathbb{C}) \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ K_0(\mathbb{C}) & \xleftarrow[K_0(f)]{} & K_0(D_n) & \xleftarrow[K_0(\iota)]{} & K_0(SM_n(\mathbb{C})) \end{array}$$

We already know that $K_1(\mathbb{C}) = 0$ and $K_0(\mathbb{C}) \cong \mathbb{Z}$. Moreover $K_0(SM_n(\mathbb{C})) \cong K_1(M_n(\mathbb{C})) = 0$ and $K_1(SM_n(\mathbb{C})) \cong K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$. Our six-term exact sequence becomes

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{K_1(\iota)} & K_1(D_n) & \xrightarrow{K_1(f)} & 0 \\ {\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\ \mathbb{Z} & \xleftarrow[K_0(f)]{} & K_0(D_n) & \xleftarrow[K_0(\iota)]{} & 0 \end{array}$$

Immediately, we see that the index map $\delta_1$ is the zero map. Now let us compute the exponential map $\delta_0$. First we begin by making some identifications. The unitisation $\tilde{D}_n$ can be identified with the set

$$\{f \in C([0,1], M_n(\mathbb{C})) \mid f(0), f(1) \in \mathbb{C}1_n\}$$

The unitisation $(SM_n(\mathbb{C}))^\sim$ is identified with the set

$$\{f \in C([0,1], M_n(\mathbb{C})) \mid f(0) = f(1) \in \mathbb{C}1_n\}.$$

Fix a one-dimensional projection $p \in M_n(\mathbb{C})$. Since $p$ has rank one then $[p]_0$ generates the group $K_0(M_n(\mathbb{C}))$. Define the projection loops $u_n, v_n \in \mathcal{U}((SM_n(\mathbb{C}))^\sim)$ by

$$u_n(t) = f_{1_n}(t) = \exp(2\pi i t 1_n) \qquad \text{and} \qquad v_n(t) = f_p(t) = \exp(2\pi i t p).$$

Now the definition of the Bott isomorphism $\beta : K_0(M_n(\mathbb{C})) \to K_1(SM_n(\mathbb{C}))$ tells us that

$$\beta([p]_0) = [v_n]_1 \qquad \text{and} \qquad \beta([1_n]_0) = [u_n]_1.$$

Consequently, $n[v_n]_1 = [u_n]_1$ in both $K_1(SM_n(\mathbb{C}))$ and $K_1(D_n)$. Hence $[v_n]_1$ is the generator of $K_1(SM_n(\mathbb{C}))$ and $K_1(D_n)$.

Now let $h$ be the function

$$\begin{array}{rccc} h: & [0,1] & \to & M_n(\mathbb{C}) \\ & t & \mapsto & t1_n. \end{array}$$

Then $h$ is a self-adjoint element in $D_n$ and $f(h) = h(1) = 1$ (we have sneakily identified $\mathbb{C}1_n$ with $\mathbb{C}$ here). Taking the exponential, we find that if $t \in [0,1]$ then

$$\exp(2\pi i h(t)) = \exp(2\pi i t 1_n) = u_n(t).$$

By applying Theorem 1.3 again, we have

$$\delta_0([1]_0) = -[u_n]_1 = -n[v_n]_1.$$

We deduce that $\delta_0 : \mathbb{Z} \to \mathbb{Z}$ is simply multiplication by $n$. Now return to our six-term exact sequence, which we rewrite as the exact sequence

$$0 \longrightarrow K_0(D_n) \xrightarrow{K_0(f)} \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z} \xrightarrow{K_1(\iota)} K_1(D_n) \longrightarrow 0.$$

By exactness, $\{0\} = \ker \delta_0 = \text{im } K_0(f)$. Since $K_0(f)$ is also injective then $K_0(D_n) = 0$. We also have $n\mathbb{Z} = \text{im } \delta_0 = \ker K_1(\iota)$. Since $K_1(\iota)$ is surjective then by the first isomorphism theorem,

$$K_1(D_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

**Example 2.7.** Let $\mathcal{T}$ denote the Toeplitz algebra. The Toeplitz algebra fits into the short exact sequence

$$0 \longrightarrow K \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

which in turn induces the six-term exact sequence

$$\begin{array}{ccccc} K_1(K) & \xrightarrow{K_1(\iota)} & K_1(\mathcal{T}) & \xrightarrow{K_1(\pi)} & K_1(C(S^1)) \\ {\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\ K_0(C(S^1)) & \xleftarrow[K_0(\pi)]{} & K_0(\mathcal{T}) & \xleftarrow[K_0(\iota)]{} & K_0(K) \end{array}$$

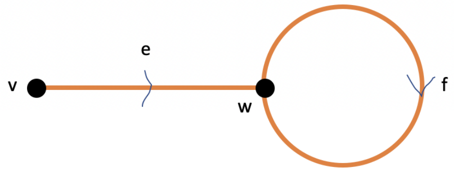
We know that $K_1(K) = 0$, $K_0(K) \cong \mathbb{Z}$ and $K_0(C(S^1)) \cong \mathbb{Z} \cong K_1(C(S^1))$. Thus the six-term exact sequence reduces to

$$\begin{array}{ccccc} 0 & \xrightarrow{K_1(\iota)} & K_1(\mathcal{T}) & \xrightarrow{K_1(\pi)} & \mathbb{Z} \\ {\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_1} \\ \mathbb{Z} & \xleftarrow[K_0(\pi)]{} & K_0(\mathcal{T}) & \xleftarrow[K_0(\iota)]{} & \mathbb{Z} \end{array}$$

The exponential map $\delta_0$ is just the zero map. By exactness, $K_0(\pi)$ is a surjective group morphism. We also observe that $K_1(\iota)$ is the zero map and as a result, $K_1(\pi)$ is an injective group morphism. So our six-term exact sequence reduces to the exact sequence

$$0 \longrightarrow K_1(\mathcal{T}) \xrightarrow{K_1(\pi)} \mathbb{Z} \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{K_0(\iota)} K_0(\mathcal{T}) \xrightarrow{K_0(\pi)} \mathbb{Z} \longrightarrow 0.$$

We claim that $K_0(\iota)$ is the zero map. To see why, we first interpret $\mathcal{T}$ as the graph C*-algebra of the following graph.




The Cuntz-Krieger relations tell us that $P_v = S_e^* S_e$, $P_w = S_f^* S_f$ and $P_w = S_e S_e^* + S_f S_f^*$. Observe that in $K_0(\mathcal{T})$,

$$[P_v]_0 + [P_w]_0 = [S_e^* S_e]_0 + [S_f^* S_f]_0 = [S_e S_e^*]_0 + [S_f S_f^*]_0 = [P_w]_0.$$

Therefore $[P_v]_0 = 0$ in the group $K_0(\mathcal{T})$. The graph C*-algebra associated to the single vertex $v$ is simply $\mathbb{C}$. Moreover, it is a full corner in the ideal $K$ (see [Rae05, Theorem 4.9]). Hence the group $K_0(K)$ is generated by the equivalence class of the rank one projection $[P_v]_0$. We also have $K_0(\iota)([P_v]_0) = 0$. Therefore $K_0(\iota)$ is the zero map.

By exactness, $\operatorname{im} \delta_1 = \ker K_0(\iota) = \mathbb{Z}$. So $\delta_1$ is a surjective group morphism from $\mathbb{Z}$ to $\mathbb{Z}$. Reasoning in the same way as Example 2.5, there are only two such surjective group morphisms ($id_{\mathbb{Z}}$ and $-id_{\mathbb{Z}}$). Both are isomorphisms. Therefore the index map $\delta_1$ is a group isomorphism.

By exactness, $\operatorname{im} K_1(\pi) = \ker \delta_1 = \{0\}$. Since $K_1(\pi)$ is injective then by the first isomorphism theorem, $K_1(\mathcal{T}) = 0$. Using exactness again, $\operatorname{im} K_0(\iota) = \ker K_1(\pi) = \{0\}$. We deduce that $K_1(\pi)$ is a group isomorphism and consequently, $K_0(\mathcal{T}) \cong \mathbb{Z}$.

**Example 2.8.** Let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$:

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Let $\mathbb{T}$ denote the boundary of $\mathbb{D}$ (this is just the circle $S^1$). Identify the open unit disk $\mathbb{D} \backslash \mathbb{T}$ with $\mathbb{R}^2$. Then we have the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \xrightarrow{\varphi} C(\mathbb{D}) \xrightarrow{\psi} C(\mathbb{T}) \longrightarrow 0.$$

where $\psi$ is the restriction map. This induces the six-term exact sequence

$$\begin{array}{ccccc} K_1(C_0(\mathbb{R}^2)) & \xrightarrow{K_1(\varphi)} & K_1(C(\mathbb{D})) & \xrightarrow{K_1(\psi)} & K_1(C(\mathbb{T})) \\ {\scriptstyle \delta_0}\big\uparrow & & & & \big\downarrow{\scriptstyle \delta_1} \\ K_0(C(\mathbb{T})) & \xleftarrow[K_0(\psi)]{} & K_0(C(\mathbb{D})) & \xleftarrow[K_0(\varphi)]{} & K_0(C_0(\mathbb{R}^2)) \end{array}$$

We know that $K_1(C_0(\mathbb{R}^2)) = 0$, $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$ and both $K_0(C(\mathbb{T}))$ and $K_1(C(\mathbb{T}))$ are isomorphic to $\mathbb{Z}$. Our six-term exact sequence reduces to

$$\begin{array}{ccccc} 0 & \xrightarrow{K_1(\varphi)} & K_1(C(\mathbb{D})) & \xrightarrow{K_1(\psi)} & \mathbb{Z} \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ \mathbb{Z} & \xleftarrow[K_0(\psi)]{} & K_0(C(\mathbb{D})) & \xleftarrow[K_0(\varphi)]{} & \mathbb{Z} \end{array}$$

The exponential map $\delta_0$ is the zero map. The group morphism $K_1(\varphi)$ is also a zero map. By exactness, $K_1(\psi)$ is an injective group morphism and $K_0(\psi)$ is a surjective group morphism.

We claim that $K_0(\psi)$ is actually an isomorphism. To see why this is the case, we will first show that $K_0(C(\mathbb{D})) \cong \mathbb{Z}$. This is due to [RLL00, Example 3.3.5], which we state and prove below as the following theorem.

**Theorem 2.1.** *Let $X$ be a compact connected Hausdorff space and $Tr$ be the standard trace on $M_n(\mathbb{C})$. There exists a surjective group homomorphism*

$$\begin{array}{rccc} \dim : & K_0(C(X)) & \to & \mathbb{Z} \\ & [p]_0 & \mapsto & Tr(p(x)) \end{array} \tag{2}$$

*where $p \in \mathcal{P}_\infty(C(X))$ and $x \in X$ is arbitrary.*

*Proof.* Assume that $X$ is a compact connected Hausdorff space.

To show: (a) If $p \in \mathcal{P}_\infty(C(X))$ then $Tr(p(x))$ is independent of $x \in X$.

(a) Assume that $p \in \mathcal{P}_\infty(C(X))$ so that $p$ is a projection in $M_n(C(X))$. The function $x \mapsto Tr(p(x))$ is an element of $C(X, \mathbb{Z})$ and is thus, a locally constant function. Note that $Tr(p(x)) \in \mathbb{Z}$ because $p$ is a projection. Since $X$ is connected then $x \mapsto Tr(p(x))$ is constant. Thus, the quantity $Tr(p(x))$ is independent of $x \in X$.

Now, if $x \in X$ then we have a trace on $C(X)$ defined by the evaluation map

$$\begin{array}{rccc} ev_x : & C(X) & \to & \mathbb{C} \\ & f & \mapsto & f(x). \end{array}$$

One can check that the conditions of [RLL00, Proposition 3.1.8] (universal property of $K_0$) are satisfied by $ev_x$. Applying the universal property, we obtain a unique group homomorphism $K_0(ev_x) : K_0(C(X)) \to \mathbb{C}$ such that if $p \in \mathcal{P}_\infty(C(X))$ then

$$K_0(ev_x)([p]_0) = ev_x(p).$$

We remark that if $n \in \mathbb{Z}_{>1}$ and $p \in \mathcal{P}_n(C(X))$ then $ev_x(p) = Tr(p(x)) \in \mathbb{Z}$. This is by convention on how traces are extended to matrix algebras (see [RLL00, Section 3.3.1]). To see that $K_0(ev_x)$ is surjective, note that $K_0(ev_x)([1_{C(X)}]_0) = 1$.

Hence, $K_0(ev_x)$ is the desired surjective group morphism. By part (a), $K_0(ev_x)$ does not depend on the choice of $x \in X$. □

**Corollary 2.2.** *Let $X$ be a contractible connected compact Hausdorff space. The surjective group homomorphism* dim *in equation* (2) *is a group isomorphism.*

*Proof.* Assume that $X$ is a contractible connected compact Hausdorff space. By contractibility, there exist $x_0 \in X$ and a continuous function $\alpha : [0,1] \times X \to X$ such that if $x \in X$ then

$$\alpha(1,x) = x \qquad \text{and} \qquad \alpha(0,x) = x_0.$$

If $t \in [0,1]$ then define the *-homomorphism

$$\begin{array}{rccc} \varphi_t : & C(X) & \to & C(X) \\ & f & \mapsto & \big(x \mapsto f(\alpha(t,x))\big) \end{array}$$

If $f \in C(X)$ then the map $t \mapsto \varphi_t(f)$ is continuous. Also, if $f \in C(X)$ and $x \in X$ then $\varphi_0(f)(x) = f(x_0)$ and $\varphi_1 = id_{C(X)}$. This shows that $\varphi_0$ is homotopic to the identity map $id_{C(X)}$. We denote this by $\varphi_0 \sim_h id_{C(X)}$.

Now let $ev_{x_0} : C(X) \to \mathbb{C}$ be the evaluation map at $x_0$. Define

$$\begin{array}{rccc} \gamma : & \mathbb{C} & \to & C(X) \\ & \lambda & \mapsto & \lambda 1. \end{array}$$

Then $ev_{x_0} \circ \gamma = id_{\mathbb{C}}$ and $\gamma \circ ev_{x_0} = \varphi_0 \sim_h id_{C(X)}$. We obtain a homotopy $\gamma \circ ev_{x_0} : C(X) \to C(X)$. Recalling the definition of dim from equation (2),

$$\dim = K_0(Tr) \circ K_0(ev_{x_0})$$

where $Tr$ is the standard trace on $M_n(\mathbb{C})$. By the homotopy invariance of $K_0$ ([RLL00, Proposition 3.2.6]),$K_0(ev_{x_0})$ is a group isomorphism. We also know from [RLL00, Example 3.3.2] that $K_0(Tr)$ is a group isomorphism from $K_0(M_n(\mathbb{C}))$ to $\mathbb{Z}$. So dim must be a group isomorphism as required. □

Applying Corollary 2.2, we find that $K_0(C(\mathbb{D})) \cong \mathbb{Z}$ and $K_0(\psi)$ is a surjective group morphism from $\mathbb{Z}$ to $\mathbb{Z}$. Repeating the same argument used in Example 2.5 and Example 2.7, we find that $K_0(\psi)$ is a group isomorphism. By exactness, im $K_0(\varphi) = \ker K_0(\psi) = \{0\}$ and hence, $K_0(\varphi)$ is the zero map.

By exactness again, im $\delta_1 = \ker K_0(\varphi) = \mathbb{Z}$. Therefore the index map $\delta_1$ is a surjective group morphism from $\mathbb{Z}$ to $\mathbb{Z}$ and hence, is also an isomorphism. Using exactness again, im $K_1(\psi) = \ker \delta_1 = \{0\}$ and since $K_1(\psi)$ is injective then $K_1(C(\mathbb{D})) = 0$ by the first isomorphism theorem.

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