Matrices invariant under Λ^2

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0.1 Motivation

Let $A, B \in M_{n\times n}(\mathbb{C})$. It is well-known that $\det(A+B) \neq \det(A) + \det(B)$. That is, the determinant map is not linear. More generally for all $k \in \{2, \dots, n\}$, $\Lambda^k(A+B) \neq \Lambda^k(A) + \Lambda^k(B)$. However, linearity is satisfied in the case where $k = 1$ because $\Lambda^1(A) = A$.

One of the fundamental techniques to studying such maps is to study its directional derivative. The directional derivative itself is a linear map and gives rise to the fundamental construct of a Lie algebra. For instance, the determinant map $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ admits a directional derivative at the matrix $A \in M_{n\times n}(\mathbb{R})$ in the direction of $B \in M_{n\times n}(\mathbb{R})$:

$$D\det_A(B) = \lim_{t\to 0} \frac{\det(A+tB) - \det(A)}{t}.$$

This linear map is surjective whenever $A \in SL_n(\mathbb{R})$. Thus, the regular value theorem tells us that the tangent space of $SL_n(\mathbb{R})$ at the identity matrix $I_n \in M_{n\times n}(\mathbb{R})$, denoted by $T_{I_n}SL_n(\mathbb{R})$, is the kernel of $D\det_{I_n}$, which is the real Lie algebra $\mathfrak{sl}_n(\mathbb{R})$.

In this short paper, we aim to apply such a technique to analyse wedge product matrices, which are defined in [Cha22, Chapter 1]. We will focus on a specific case, where concrete computations are not too taxing.

0.2 The analysis

We will work with 3×3 matrices with complex entries ($M_{3\times 3}(\mathbb{C})$). If $A \in M_{3\times 3}(\mathbb{C})$, then we recall that the matrix $\Lambda^2(A)$ consists of all the 2×2 minors of A. We can express this map explicitly as follows:

$$\begin{aligned}\Lambda^2 : \quad M_{3\times 3}(\mathbb{C}) \quad &\to \quad M_{3\times 3}(\mathbb{C}) \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} ae-bd & af-cd & bf-ec \\ ah-bg & ai-cg & bi-ch \\ dh-eg & di-fg & ei-fh \end{pmatrix}.\end{aligned}$$

It is notable that in the three dimensional case, the map Λ^2 satisfies $\Lambda^2(\Lambda^2(A)) = \det(A)A$ for all $A \in M_{3\times 3}(\mathbb{C})$. We can think of Λ^2 as a map from $\mathbb{C}^9$ to $\mathbb{C}^9$, where $\mathbb{C}^9$ has the Euclidean topology. Since each component function of Λ^2 are polynomials of the original inputs, Λ^2 must be a smooth function, since polynomials are smooth. Hence, it makes sense to talk about derivatives of Λ^2.

More specifically, the map we are interested in is the directional derivative of Λ^2 in the direction of the identity I_3.

$$D^2 : M_{3\times 3}(\mathbb{C}) \to M_{3\times 3}(\mathbb{C})$$

$$A \mapsto \lim_{t\to 0} \frac{\Lambda^2(A + tI_3) - \Lambda^2(A)}{t}.$$

There are two different methods of understanding the map D^2, which we will outline below:

1. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in M_{3\times 3}(\mathbb{C}).$$

   Then, we can compute $D^2(A)$ directly to obtain

$$D^2 \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+e & f & -c \\ h & a+i & b \\ -g & d & e+i \end{pmatrix}. \tag{1}$$

   From this direct computation, we observe that D^2 is indeed a linear map.

2. Since $\mathbb{C}$ is an algebraically closed field, A can be expressed in its Jordan normal form. That is, $A = PJP^{-1}$, where $P \in GL_3(\mathbb{C})$ and

$$J = \begin{pmatrix} \lambda_1(A) & x & y \\ & \lambda_2(A) & z \\ & & \lambda_3(A) \end{pmatrix}.$$

   Here, $\lambda_1(A), \lambda_2(A)$ and $\lambda_3(A)$ are the eigenvalues of A. Using this decomposition, we compute $D^2(A)$ as follows:

$$
\begin{aligned}
D^2(A) &= \lim_{t\to 0}\frac{\Lambda^2(A+tI_n)-\Lambda^2(A)}{t}\\
&= \lim_{t\to 0}\frac{\Lambda^2(PJP^{-1}+tI_n)-\Lambda^2(PJP^{-1})}{t}\\
&= \lim_{t\to 0}\frac{\Lambda^2(P)\Lambda^2(J+tI_n)(\Lambda^2(P))^{-1}-\Lambda^2(P)\Lambda^2(J)(\Lambda^2(P))^{-1}}{t}\\
&= \Lambda^2(P)\Big(\lim_{t\to 0}\frac{\Lambda^2(J+tI_n)-\Lambda^2(J)}{t}\Big)(\Lambda^2(P))^{-1}\\
&= \Lambda^2(P)\begin{pmatrix}\lambda_1(A)+\lambda_2(A) & z & -y\\ & \lambda_1(A)+\lambda_3(A) & x\\ & & \lambda_2(A)+\lambda_3(A)\end{pmatrix}(\Lambda^2(P))^{-1}.
\end{aligned}
$$

Let us study some more properties of D^2.

Proposition 0.2.1. *Let $A \in M_{3\times 3}(\mathbb{C})$. Then,*

(a) $D^2(D^2(A)) = A + Tr(A)I_3$.

(b) D^2 is a bijective map.

Proof. Assume that

$$
A = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} \in M_{3\times 3}(\mathbb{C}).
$$

For part (a), we can compute directly using (1) to obtain

$$
D^2(D^2(A)) = \begin{pmatrix} 2a+e+i & b & c\\ d & a+2e+i & f\\ g & h & a+e+2i\end{pmatrix} = A + (a+e+i)I_3.
$$

We will also use equation (1) to prove part (b) of the proposition.

To show: (a) D^2 is injective.

(b) D^2 is surjective.

(a) Assume that $A \in \ker D^2$. Then, by (1),

$$D^2(A) = \begin{pmatrix} a+e & f & -c \\ h & a+i & b \\ -g & d & e+i \end{pmatrix} = 0.$$

By comparing the entries of both matrices, we find that all of a, b, c, d, e, f, g, h and i are all equal to zero. So, $\ker D^2 = \{0\}$ and thus, D^2 is injective.

(b) We will show that there exists $B \in M_{3\times3}(\mathbb{C})$ such that $D^2(B) = A$. Define

$$B = \begin{pmatrix} \frac{1}{2}(a+e-i) & f & -c \\ h & \frac{1}{2}(a-e+i) & b \\ -g & d & \frac{1}{2}(-a+e+i) \end{pmatrix}.$$

A quick calculation shows that the matrix B satisfies $D^2(B) = A$. Hence, D^2 is surjective.

By combining parts (a) and (b) of the proof, we find that D^2 is a bijective map. □

The first hint of Lie algebras in this paper emerges as a result of 0.2.1. In particular, if $A \in \mathfrak{sl}_3(\mathbb{C})$ then $D^2(D^2(A)) = A$. This identity suggests that we investigate matrices which are "fixed points" of D^2. In other words, when does a matrix $B \in M_{3\times3}(\mathbb{C})$ satisfy $D^2(B) = B$?

Again, we let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in M_{3\times3}(\mathbb{C}).$$

If $D^2(A) = A$, then by (1),

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+e & f & -c \\ h & a+i & b \\ -g & d & e+i \end{pmatrix}.$$

This means that $b = f$, $d = h$, $c = g = 0$, $a = -i$ and $e = 0$. Therefore, the set of matrices preserved by D^2 is given by

$$\mathfrak{d}_2 = \left\{ \begin{pmatrix} a & b & 0 \\ d & 0 & b \\ 0 & d & -a \end{pmatrix} \mid a, b, d \in \mathbb{C} \right\}.$$

What structure does this set have? It certainly has the structure of a $\mathbb{C}$-vector space. However, it is not closed under matrix multiplication because if we let

$$A_1 = \begin{pmatrix} a_1 & b_1 & 0 \\ d_1 & 0 & b_1 \\ 0 & d_1 & -a_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} a_2 & b_2 & 0 \\ d_2 & 0 & b_2 \\ 0 & d_2 & -a_2 \end{pmatrix},$$

then

$$A_1A_2 = \begin{pmatrix} a_1a_2 + b_1d_2 & a_1b_2 & b_1b_2 \\ d_1a_2 & d_1b_2 + b_1d_2 & -b_1a_2 \\ d_1d_2 & -a_1d_2 & d_1b_2 + a_1a_2 \end{pmatrix} \notin \mathfrak{d}_2.$$

However,

$$[A_1, A_2] = A_1A_2 - A_2A_1 = \begin{pmatrix} b_1d_2 - b_2d_1 & a_1b_2 - a_2b_1 & 0 \\ d_1a_2 + a_1d_2 & 0 & -b_1a_2 + b_2a_1 \\ 0 & -a_1d_2 + a_2d_1 & d_1b_2 - d_2b_1 \end{pmatrix}$$

which is an element of $\mathfrak{d}_2$. We stress the importance of this computation with the following theorem

Theorem 0.2.2. *Let*

$$\mathfrak{d}_2 = \left\{ \begin{pmatrix} a & b & 0 \\ d & 0 & b \\ 0 & d & -a \end{pmatrix} \mid a, b, d \in \mathbb{C} \right\}.$$

Then, $\mathfrak{d}_2$ is a complex Lie algebra, with Lie bracket $[A, B] = AB - BA$ for all $A, B \in \mathfrak{d}_2$. Furthermore, it is a Lie subalgebra of $\mathfrak{sl}_3(\mathbb{C})$.

Let us examine some properties of $\mathfrak{d}_2$. First, we observe that it has basis given by

$$L_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, L_2 = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix} \text{ and } L_3 = \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix}.$$

The associated commutator relations are

$$[L_1, L_2] = L_2, [L_1, L_3] = -L_3 \text{ and } [L_2, L_3] = L_1.$$

We know that a matrix Lie group and its associated Lie algebra is connected by the exponential map. For example, if $A \in \mathfrak{gl}_3(\mathbb{C})$, then for all

$t \in \mathbb{C}$, $\exp(tA) \in GL_3(\mathbb{C})$. By applying the exponential map to the basis elements of $\mathfrak{d}_2$, we obtain for all $t \in \mathbb{C}$,

$$\exp(tL_1) = \begin{pmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \end{pmatrix}, \exp(tL_2) = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{pmatrix} \text{ and } \exp(tL_3) = \begin{pmatrix} 1 & & \\ t & 1 & \\ \frac{t^2}{2} & t & 1 \end{pmatrix}.$$

The most important property of these matrices is that they do not change when the map Λ^2 is applied to it. Since $L_1, L_2, L_3 \in \mathfrak{sl}_3(\mathbb{C})$, the above matrix exponentials are expected to be elements of $SL_3(\mathbb{C})$, which is true. The following theorem generalises this property to the entirety of $\mathfrak{d}_2$.

Theorem 0.2.3. *Let $A \in \mathfrak{d}_2$. Then, for all $t \in \mathbb{C}$, $\Lambda^2(\exp(tA)) = \exp(tA)$.*

Proof. Assume that

$$A = \begin{pmatrix} a & b & 0 \\ d & 0 & b \\ 0 & d & -a \end{pmatrix} \in \mathfrak{d}_2.$$

We note that A is diagonalisable with eigenvalues given by $0, \pm\sqrt{a^2 + 2bd}$. So, we can write

$$A = P \begin{pmatrix} \sqrt{a^2+2bd} & & \\ & 0 & \\ & & -\sqrt{a^2+2bd} \end{pmatrix} P^{-1}$$

where $P \in GL_3(\mathbb{C})$. Since, $D^2(A) = A$, we can use our second characterisation of D^2 to show that

$$\Lambda^2(P) \begin{pmatrix} \sqrt{a^2+2bd} & & \\ & 0 & \\ & & -\sqrt{a^2+2bd} \end{pmatrix} (\Lambda^2(P))^{-1} = P \begin{pmatrix} \sqrt{a^2+2bd} & & \\ & 0 & \\ & & -\sqrt{a^2+2bd} \end{pmatrix} P^{-1}.$$

So, for all $t \in \mathbb{C}$, we have two different expressions for the matrix exponential $\exp(tA)$, which are

$$\exp(tA) = \Lambda^2(P) \begin{pmatrix} \exp(t\sqrt{a^2+2bd}) & & \\ & 1 & \\ & & \exp(-t\sqrt{a^2+2bd}) \end{pmatrix} (\Lambda^2(P))^{-1} \tag{2}$$

and

$$\exp(tA) = P \begin{pmatrix} \exp(t\sqrt{a^2+2bd}) & & \\ & 1 & \\ & & \exp(-t\sqrt{a^2+2bd}) \end{pmatrix} P^{-1}. \qquad (3)$$

Let us set

$$D = \begin{pmatrix} \exp(t\sqrt{a^2+2bd}) & & \\ & 1 & \\ & & \exp(-t\sqrt{a^2+2bd}) \end{pmatrix}.$$

Observe that $\Lambda^2(D) = D$. Using (3), we can compute $\Lambda^2(\exp(tA))$ as

$$\begin{aligned} \Lambda^2(\exp(tA)) &= \Lambda^2(PDP^{-1}) \quad (3) \\ &= \Lambda^2(P)\Lambda^2(D)(\Lambda^2(P))^{-1} \\ &= \Lambda^2(P)D(\Lambda^2(P))^{-1} \\ &= \exp(tA). \quad (2) \end{aligned}$$

This completes the proof.

□

Thus, theorem 0.2.3 tells us that the matrix Lie group associated with the Lie algebra $\mathfrak{d}_2$ is

$$Fix(\Lambda^2) = \{A \in SL_3(\mathbb{C}) \mid \Lambda^2(A) = A\}.$$

The example below allows us to give plenty of examples of matrices in $Fix(\Lambda^2)$.

Example 0.2.1. We will begin with a matrix in the Lie algebra $\mathfrak{d}_2$:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix} \in \mathfrak{d}_2.$$

The matrix A has the diagonalisation $A = PDP^{-1}$, where

$$P = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ -2 & -1 & -2 \\ -1 & -2 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 3 & & \\ & 0 & \\ & & -3 \end{pmatrix}.$$

Theorem 0.2.3 tells us that if $t \in \mathbb{C}$ then
$\exp(tA) = P\exp(D)P^{-1} \in Fix(\Lambda^2)$. Written out explicitly,

$$\exp(tA) = P \begin{pmatrix} e^{3t} & & \\ & 1 & \\ & & e^{-3t} \end{pmatrix} P^{-1}.$$

Let us substitute some explicit values for t to obtain examples of matrices in $Fix(\Lambda^2)$:

1. $t = \frac{1}{3}\ln 2$:

$$P \begin{pmatrix} 2 & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix} P^{-1} = \frac{1}{18} \begin{pmatrix} 25 & 10 & 2 \\ 10 & 22 & 8 \\ 2 & 8 & 16 \end{pmatrix}.$$

2. $t = \frac{1}{3}\ln 7$:

$$P \begin{pmatrix} 7 & & \\ & 1 & \\ & & \frac{1}{7} \end{pmatrix} P^{-1} = \frac{1}{7} \begin{pmatrix} 25 & 20 & 8 \\ 20 & 23 & 12 \\ 8 & 12 & 9 \end{pmatrix}.$$

3. $t = \frac{i\pi}{6}$.

$$P \begin{pmatrix} i & & \\ & 1 & \\ & & -i \end{pmatrix} P^{-1} = \frac{1}{9} \begin{pmatrix} 4+3i & -2+6i & -4 \\ -2+6i & 1 & 2+6i \\ -4 & 2+6i & 4-3i \end{pmatrix}.$$

The reader is invited to check that the above matrices are indeed invariant under the map Λ^2.

The generators L_1, L_2 and L_3 of $\mathfrak{d}_2$ bear an uncanny resemblance to the generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ (credit goes to Arun for this observation). Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

be the generators of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 0.2.4 ($\mathfrak{d}_2$ Isomorphism)**.** *We have a Lie algebra isomorphism given by*

$$\phi : \mathfrak{d}_2 \to \mathfrak{sl}_2(\mathbb{C})$$

$$L_1 \mapsto \frac{1}{2}e + \frac{1}{2}f$$

$$L_2 \mapsto \frac{1}{2}h - \frac{1}{2}e + \frac{1}{2}f$$

$$L_3 \mapsto \frac{1}{4}h + \frac{1}{4}e - \frac{1}{4}f$$

Proof. Assume that ϕ is defined as above. First, we observe that the set $\{\frac{1}{2}(e+f), \frac{1}{2}(h-e+f), \frac{1}{4}(h+e-f)\}$ is a basis for $\mathfrak{sl}_2(\mathbb{C})$ because

1. $h = (\frac{1}{2}(h-e+f)) + 2(\frac{1}{4}(h+e-f))$
2. $e = (\frac{1}{2}(e+f)) - \frac{1}{2}(\frac{1}{2}(h-e+f)) + (\frac{1}{4}(h+e-f))$
3. $f = (\frac{1}{2}(e+f)) + \frac{1}{2}(\frac{1}{2}(h-e+f)) - (\frac{1}{4}(h+e-f))$

and

$$\begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = \frac{1}{4} \neq 0.$$

Since ϕ is a map between the basis elements of $\mathfrak{d}_2$ and $\mathfrak{sl}_2(\mathbb{C})$, we can extend it by linearity to all of $\mathfrak{d}_2$ and $\mathfrak{sl}_2(\mathbb{C})$. So, ϕ is a bijective linear morphism. It remains to show that ϕ preserves the Lie bracket. To see this, we compute

$$\begin{aligned} [\phi(L_1), \phi(L_2)] &= [\frac{1}{2}e + \frac{1}{2}f, \frac{1}{2}h - \frac{1}{2}e + \frac{1}{2}f] \\ &= \frac{1}{4}[e,h] + \frac{1}{2}[e,f] + \frac{1}{4}[f,h] \\ &= -\frac{1}{2}e + \frac{1}{2}h + \frac{1}{2}f \\ &= \phi(L_2) \\ &= \phi([L_1, L_2]). \end{aligned}$$

$$\begin{aligned} [\phi(L_1), \phi(L_3)] &= [\frac{1}{2}e + \frac{1}{2}f, \frac{1}{4}h + \frac{1}{4}e - \frac{1}{4}f] \\ &= \frac{1}{8}[e,h] - \frac{1}{4}[e,f] + \frac{1}{8}[f,h] \\ &= -\frac{1}{4}e - \frac{1}{4}h + \frac{1}{4}f \\ &= -\phi(L_3) \\ &= \phi([L_1, L_3]). \end{aligned}$$

$$
\begin{aligned}
[\phi(L_2), \phi(L_3)] &= [\frac{1}{2}h - \frac{1}{2}e + \frac{1}{2}f, \frac{1}{4}h + \frac{1}{4}e - \frac{1}{4}f] \\
&= -\frac{1}{4}[e, h] + \frac{1}{4}[f, h] \\
&= \frac{1}{2}e + \frac{1}{2}f \\
&= \phi(L_1) \\
&= \phi([L_2, L_3]).
\end{aligned}
$$

Hence, ϕ is a Lie algebra isomorphism.

□

Bibliography

[Cha22] Y. Chan. *Wedge product matrices and applications*, University of Melbourne, October 7th 2022.