The Ising Model

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Contents

0.1 Introduction

The Ising model is a spin model designed to describe the phenomenon of ferromagnetism (and the related antiferromagnetism). It was first solved in the one-dimensional case by Ising in the 20^{th} century, who mistakenly believed that it could not be solved in two dimensions. Lars Onsager proceeded to solve it in two-dimensions. So far, the Ising model is known to be exactly solvable in only one and two dimensions.

The fundamental technique to the solution of the Ising model is the *transfer matrix technique*, which we will attempt to describe in detail within these notes.

0.2 The one-dimensional case

Consider a line of n points. The set of states of our system, denoted by $\mathcal{C}$, is given by

$$\mathcal{C} = \{\phi : \mathbb{Z}/n\mathbb{Z} \to \{-1, 1\}\}.$$

This is the set of functions from $\mathbb{Z}/n\mathbb{Z}$ to $\{-1, 1\}$. The set $\{-1, 1\}$ represents the "spin" of each point. One can think of -1 representing a down spin and 1 representing an up spin. We select the set $\mathbb{Z}/n\mathbb{Z}$ in order to impose periodic boundary conditions on our model. That is, $s_{n+1} = s_1$ or the value of the spin at site $n + 1$ must match that of the first spin.

The most important aspect of the model however, is the energy/Hamiltonian, which is given by

$$E = -J \sum_{i=1}^{n} s_i s_{i+1} - H \sum_{i=1}^{n} s_i.$$

where s_i denotes the spin at site i and $J, H \in \mathbb{R}$. The constant J is the interaction term between each pair of spins, whereas the constant H is the magnetic field term on each individual spin.

Let $\beta = 1/T$ be the reciprocal temperature. Then, the partition function of this Ising model is given by

$$Z = \sum_{s \in \mathcal{C}} e^{-\beta E(s)} = \sum_{s \in \mathcal{C}} e^{K \sum_{i=1}^{n} s_i s_{i+1} + B \sum_{i=1}^{n} s_i}$$

Here, $K = \beta J$ and $B = \beta H$. Essentially, we are summing over all the possible states of the system. Explicitly, we can write

$$Z = \sum_{s_1 \in \{-1,1\}} \cdots \sum_{s_n \in \{-1,1\}} e^{K \sum_{i=1}^{n} s_i s_{i+1} + B \sum_{i=1}^{n} s_i}.$$

In order for the transfer matrix technique to work, one must rewrite the summand of the partition function in a more symmetric manner, so that each pair of neighbouring spins is represented equally. Keeping this in mind, we obtain

$$Z = \sum_{s_1 \in \{-1,1\}} \cdots \sum_{s_n \in \{-1,1\}} e^{K s_1 s_2} e^{\frac{B}{2}(s_1 + s_2)} \dots e^{K s_n s_1} e^{\frac{B}{2}(s_n + s_1)}.$$

The key idea here is to recognise the above sum as looking somewhat like the multiplication of matrices. In fact, the transfer matrix technique makes this a reality. Define

$$T = \begin{pmatrix} e^{K+B} & e^{-K} \\ e^{-K} & e^{K-B} \end{pmatrix}.$$

We can think of this matrix as a 2×2 matrix indexed by each pair of neighbouring spins s_i, s_{i+1} such that if

$$t(s_i, s_{i+1}) = e^{K s_i s_{i+1}} e^{\frac{B}{2}(s_i + s_{i+1})}$$

then

$$T = \begin{pmatrix} t(1,1) & t(1,-1) \\ t(-1,1) & t(-1,-1) \end{pmatrix}.$$

As a result, the partition function simplifies immensely to give a trace:

$$Z = \sum_{s_1 \in \{-1,1\}} \cdots \sum_{s_n \in \{-1,1\}} T_{s_1,s_2} T_{s_2,s_3} \dots T_{s_n,s_1} = Tr(T^n).$$

Now, we are left to compute the trace. The first observation is that T has two eigenvalues λ_+ and λ_-, given by

$$\lambda_\pm = e^K \cosh(B) \pm \sqrt{e^{2K} \sinh^2(B) + e^{-2K}}.$$

By diagonalising T, we are able to compute the partition function as $Z = Tr(T^n) = \lambda_+^n + \lambda_-^n$.

0.3 Important quantities of the 1D Ising model

0.3.1 Free energy and phase transitions

With the partition function in hand, we can now compute various important quantities of the 1D Ising model, in order to understand it better.

First up is the free energy per site, which is defined by

$$f = \frac{\log Z}{n}.$$

Most importantly, the free energy is able to describe whether our model has a *phase transition* — a discontinuity in the free energy represents a phase

transition which occurs at a critical temperature T_c. We will see soon that as the size of the system tends to infinity, there is no such discontinuity in the free energy and hence, no phase transition in the 1D Ising model.

The key observation to the next computation is that $\lambda_+ > \lambda_-$ and so, as $n \to \infty$,

$$\lim_{n\to\infty} \lambda_+^n + \lambda_-^n = \lim_{n\to\infty} \lambda_+^n (1 + \frac{\lambda_-^n}{\lambda_+^n}) \approx \lambda_+^n.$$

Hence, as the size of our system grows very large, the free energy per site becomes

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log Z}{n} &= \lim_{n\to\infty} \frac{\log(\lambda_+^n + \lambda_-^n)}{n} \\
&\approx \lim_{n\to\infty} \frac{\log(\lambda_+^n)}{n} \\
&= \lim_{n\to\infty} \log(\lambda_+) \\
&= \log(\Lambda_+).
\end{aligned}$$

Note that $\log(\Lambda_+)$ is a continuous function of β, since the logarithm, square root and hyperbolic functions are all continuous functions. In particular, there are no discontinuities associated with the variable $\beta \in \mathbb{R}_{>0}$. Hence, there cannot be any phase transitions in the one dimensional Ising model.

0.3.2 Average spin and spin correlation

Assume that $i, j \in \{1, \ldots, n\}$ with $i \neq j$. By definition, the average spin $\langle s_i \rangle$ is equal to

$$\begin{aligned}
\langle s_i \rangle &= \frac{1}{Z} \sum_{s\in\mathcal{C}} s_i e^{-\beta E(s)} \\
&= \frac{1}{Z} \sum_{s_1\in\{-1,1\}} \cdots \sum_{s_n\in\{-1,1\}} s_i e^{K s_1 s_2} e^{\frac{B}{2}(s_1+s_2)} \ldots e^{K s_n s_1} e^{\frac{B}{2}(s_n+s_1)} \\
&= \frac{1}{Z} \sum_{s_1\in\{-1,1\}} \cdots \sum_{s_n\in\{-1,1\}} T_{s_1,s_2} \ldots T_{s_{i-1},s_i} s_i T_{s_i,s_{i+1}} \ldots T_{s_n,s_1}.
\end{aligned}$$

We want to express the i^{th} spin s_i with a 2×2 matrix. To this end, we define

$$\sigma^z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

This is a matrix indexed by spins s_i and s_{i+1} so that the element $(\sigma^z)_{s_i,s_{i+1}} = s_i\delta_{s_i,s_{i+1}}$, where δ is the Kronecker delta. Notice that $(\sigma^z)_{s_i,s_{i+1}} \neq 0$ whenever $s_i = s_{i+1}$. Since we are only summing over $s_i \in \{-1, 1\}$, we have

$$\langle s_i \rangle = \frac{1}{Z} \sum_{s_1 \in \{-1,1\}} \cdots \sum_{s_n \in \{-1,1\}} T_{s_1,s_2} \ldots T_{s_{i-1},s_i} (\sigma^z)_{s_i,s_i} T_{s_i,s_{i+1}} \ldots T_{s_n,s_1}.$$

By using the transfer matrix method, we can do the matrix multiplications and express the above sum as a trace, revealing that

$$\langle s_i \rangle = \frac{1}{Z} Tr(T^i \sigma^z T^{n-i}).$$

For the spin correlation factor $\langle s_i s_j \rangle$, the process is very similar to computing the average spin because

$$\langle s_i s_j \rangle = \frac{1}{Z} \sum_{s \in \mathcal{C}} s_i s_j e^{-\beta E(s)}.$$

By going through the similar derivation, we obtain

$$\langle s_i s_j \rangle = \frac{1}{Z} Tr(T^i \sigma^z T^{j-i} \sigma^z T^{n-j}).$$

0.3.3 The magnetic field free regime

In this section, we set $B = 0$. Our eigenvalues λ_+ and λ_- become

$$\lambda_+ = 2\cosh K \text{ and } \lambda_- = 2\sinh K.$$

Hence, the partition function for the Ising model without the magnetic field term B is

$$Z = (2\cosh K)^n + (2\sinh K)^n.$$

We will recalculate the average spin and spin correlation factors. Let $i, j \in \{1, \ldots, L\}$ with $j \geq i$. Then, the average spin is given by

$$\begin{aligned}\langle s_i \rangle &= \frac{1}{Z} Tr(T^i \sigma^z T^{n-i}) \\ &= \frac{1}{Z} Tr(\sigma^z T^{n-i} T^i) \\ &= \frac{1}{Z} Tr(\sigma^z T^n).\end{aligned}$$

We want to show that $Tr(\sigma^z T^n) = 0$. To see why this is the case, define the set

$$A = \{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \}.$$

Note that $T \in A$. A quick computation reveals that if $a, b, c, d \in \mathbb{R}$,

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} c & d \\ d & c \end{pmatrix} = \begin{pmatrix} ac + bd & ad + bc \\ ad + bc & ac + bd \end{pmatrix}.$$

Thus, the set A is a monoid when equipped with matrix multiplication. In particular, this means that T^n can be written in the form

$$T^n = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \in A \text{ so that } \sigma^z T^n = \begin{pmatrix} X & Y \\ -Y & -X \end{pmatrix}.$$

Therefore, $\langle s_i \rangle = Tr(\sigma^z T^n) = 0$. We expect the average spin to be zero because each lattice site has an equal probability to be an up spin or a down spin. To compute the spin correlation factor $\langle s_i s_j \rangle$, define the function

$$f : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$

$$f(u) = (e^K I_2 - e^{-K} u)^{j-i} (e^K I_2 + e^{-K} u)^{L-j+i}.$$

where I_2 is the 2×2 identity matrix. Then,

$f(I_2) = (2 \sinh K I_2)^{j-i} (2 \cosh K I_2)^{L-j+i}$ and $f(-I_2) = (2 \cosh K I_2)^{j-i} (2 \sinh K I_2)^{L-j+i}$.

Recall that

$$\sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The identity that we require is

$$
\begin{aligned}
f(\sigma^x) &= \begin{pmatrix} 2\sinh K & \\ & 2\cosh K \end{pmatrix}^{j-i} \begin{pmatrix} 2\cosh K & \\ & 2\sinh K \end{pmatrix}^{L-j+i} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2\sinh K & \\ & 2\sinh K \end{pmatrix}^{j-i} \begin{pmatrix} 2\cosh K & \\ & 2\cosh K \end{pmatrix}^{L-j+i} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\cosh K & \\ & 2\cosh K \end{pmatrix}^{j-i} \begin{pmatrix} 2\sinh K & \\ & 2\sinh K \end{pmatrix}^{L-j+i} \\
&= \frac{1}{2}(I_2 + \sigma^x) f(I_2) + \frac{1}{2}(I_2 - \sigma^x) f(-I_2).
\end{aligned}
$$

Using our expression for the spin correlation factor, we compute that

$$
\begin{aligned}
\langle s_i s_j \rangle &= \frac{1}{Z} Tr(T^i \sigma^z T^{j-i} \sigma^z T^{n-j}) \\
&= \frac{1}{Z} Tr(\sigma^z T^{j-i} \sigma^z T^{n-j+i}) \\
&= \frac{1}{Z} Tr((\sigma^z T \sigma^z)^{j-i} T^{n-j+i}) \\
&= \frac{1}{Z} Tr((e^K I_2 - e^{-K}\sigma^x)^{j-i} (e^K I_2 + e^{-K}\sigma^x)^{n-j+i}) \\
&= \frac{1}{Z} Tr(f(\sigma^x)) \\
&= \frac{1}{Z} Tr(\frac{1}{2}(I_2 + \sigma^x) f(I_2) + \frac{1}{2}(I_2 - \sigma^x) f(-I_2)) \\
&= \frac{1}{Z} [(2\sinh K)^{j-i} (2\cosh K)^{n-j+i} + (2\cosh K)^{j-i} (2\sinh K)^{n-j+i}] \\
&= \frac{1}{Z} [(\tanh K)^{j-i} (2\cosh K)^n + (2\cosh K)^n (\tanh K)^{n-j+i}] \\
&= \frac{(2\cosh K)^n}{(2\sinh K)^n + (2\cosh K)^n} ((\tanh K)^{j-i} + (\tanh K)^{n-j+i}) \\
&= \frac{(\tanh K)^{j-i} + (\tanh K)^{n-j+i}}{1 + (\tanh K)^n}.
\end{aligned}
$$

The quantities that we will compute next are quantities which are more familiar from the perspective of statistical mechanics. The free energy F is given by

$$
F = -T \log Z = -\frac{1}{\beta} \log((2\cosh \beta J)^n + (2\sinh \beta J)^n).
$$

The average energy $\langle E \rangle$ of the system is

$$\begin{aligned}
\langle E \rangle &= -\frac{\partial \log Z}{\partial \beta} \\
&= -\frac{\partial}{\partial \beta}(\log((2\cosh \beta J)^n + (2\sinh \beta J)^n)) \\
&= -\frac{n2^n J[\cosh^{n-1} \beta J \sinh \beta J + \sinh^{n-1} \beta J \cosh \beta J]}{2^n(\cosh^n \beta J + \sinh^n \beta J)} \\
&= -\frac{nJ[\cosh^{n-1} \beta J \sinh \beta J + \sinh^{n-1} \beta J \cosh \beta J]}{\cosh^n \beta J + \sinh^n \beta J}.
\end{aligned}$$

The entropy and the heat capacity are given by

$$S = \beta^2 \frac{\partial F}{\partial \beta} \text{ and } C = \beta^2 \frac{\partial^2 \log Z}{\partial \beta^2}.$$

These result in complicated expressions. The final expression we will compute is the correlation length ξ associated with the spin correlation factor $\langle s_i s_j \rangle$. Roughly speaking, the correlation length tells you how quickly the spin correlation function vanishes as the separation between two spins grows very large (or as $|j - i| \to \infty$). It takes the form of an exponential away from the potential critical temperature T_c ($\langle s_i s_j \rangle \sim \exp(-|j - i|/\xi)$), whereas at the critical temperature T_c, the correlation length $\eta \sim |T - T_c|^{-\nu}$, where ν is the **critical exponent** associated with the spin correlation.

We will make the approximation $1 << |j - i| << n$.

Since $\tanh K \in (-1, 1)$, $(\tanh K)^n \to 0$ as $n \to \infty$. Since $n >> j - i$, $(\tanh K)^{n-j+i} \to 0$ as $n \to \infty$. As a result, we have

$$\begin{aligned}
\langle s_i s_j \rangle &= \frac{(\tanh K)^{j-i} + (\tanh K)^{n-j+i}}{1 + (\tanh K)^n} \\
&\approx (\tanh K)^{j-i} \\
&= e^{\log((\tanh K)^{j-i})} \\
&= e^{-(j-i)\log(\coth K)}.
\end{aligned}$$

Therefore, the correlation length is $\xi = \frac{1}{\log(\coth K)}$, where $K > 0$. Of course, since there are no phase transitions in the 1D Ising model, there are no critical exponents to compute in this case.

0.4 The Ising model on a square lattice

0.4.1 Duality

Before we delve into the Ising model on a two-dimensional square lattice, we must discuss the important concept of **duality**, which roughly states that low temperature Ising models can be converted to high temperature Ising models and vice versa. Before Onsager's full solution to the two-dimensional Ising model, Kramers and Wannier used a duality transformation to determine the critical temperature of the Ising model.

We will follow the expositions in the references [Bax89] and [Mus10]. We begin by deriving the low temperature series expansion of the partition function of the Ising model on a $M \times M$ square lattice.

0.4.2 Low temperature partition function

Consider a $M \times M$ square lattice, where each spin can either take the values of -1 or 1 (up and down spins). Assume that we do not have any magnetic field term B. For a given configuration of spins, let r denote the number of vertical edges in which the two adjacent spins are antiparallel (different). Similarly, let s denote the number of horizontal edges in which the two adjacent spins are antiparallel. This means that there are $M - r$ and $M - s$ vertical and horizontal edges with parallel adjacent spins.

The partition function for the two-dimensional square lattice Ising model is

$$Z = \sum_{s \in \mathcal{C}} \sum_{(i,j)} \exp(K s_{i,j} s_{i+1,j} + L s_{i,j} s_{i,j+1}). \tag{1}$$

Here, K is an interaction term between the horizontal edges and L is an interaction term between the vertical edges. A configuration with r antiparallel vertical edges and s antiparallel horizontal edges will therefore contribute to the partition function

$$\exp(K((M-s)-s)+L((M-r)-r)) = \exp(K(M-2s)+L(M-2r)). \tag{2}$$

Here is the duality transformation. For a particular configuration on our square lattice, we can associated to it a *dual lattice*, where the spins are located on the faces of the square lattice. The dual lattice is shifted from the original lattice by a "half-lattice spacing" in both directions. This

means that for all edges in the original lattice, there exists exactly one edge from the dual lattice which intersects the edge from the original lattice. The rules for such an association are as follows:

1. If an edge on the original lattice is antiparallel, we say that the intersecting edge on the dual lattice is occupied and we draw a line on this edge.

2. On the other hand, if an edge on the original lattice is antiparallel, then the intersecting edge on the dual lattice is unoccupied and we do nothing.

In this manner, we obtain r occupied horizontal lines and s occupied vertical lines on the dual lattice. An important observation here is that for each site on the original lattice, there is an even number of successive spin changes between the four surrounding faces on the dual lattice. So, there is an even number of lines into each site. Hence, the lines on the dual lattice form **closed polygons**, which separate the up and down spins into two different domains.

For each closed polygon P, there are two spin configurations, with one being obtained from the other via a spin flip. Therefore, the partition function in (1) can be expressed as

$$Z = 2\exp(M(K+L))\sum_{P}\exp(-2Lr-2Ks) \tag{3}$$

where the sum is over all closed polygons in the dual lattice (i.e. over all occupied edges with an even number of lines into each site). In this case, r and s refer to the number of occupied horizontal and vertical lines respectively.

An important observation associated with (3) is that the partition function can be expressed with the geometrical quantity

$$\Phi(K,L) = \sum_{P}\exp(-2Lr-2Ks).$$

Let us compute the first few terms of the series expansion for Φ. The first term occurs when all of the spins on the dual lattice are of the same value. In this case, the values of r and s are both zero and thus, the first term in the series expansion is 1.

The second term occurs when one of the spins is (say) down, whereas the rest of them are up. In this case, $r = s = 2$ so that the occupied edges form a square around the flipped spin. Since there are N lattice sites, there are N possible choice of where the spin flip occurs. Hence, the second term in the expansion is

$$N \exp(-4L - 4K).$$

The third term originates from closed polygon rectangles, which are either 2×1 or 1×2. So, we have either $r = 4, s = 2$ or $r = 2, s = 4$ respectively. The degeneracy factor is still N (the number of lattice sites). Hence, the third term contributing to Φ is

$$N(e^{-8L-4K} + e^{-4L-8K}).$$

Therefore,

$$\Phi(K, L) = 1 + Ne^{-4L-4K} + N(e^{-8L-4K} + e^{-4L-8K}) + \dots.$$

0.4.3 High temperature partition function

We begin with the identity

$$\exp(Ks_is_j) = \cosh K + \sinh K s_i s_j \tag{4}$$

where $s_i, s_j \in \{-1, 1\}$. A similar expression also holds for L. This holds because for all lattice sites i and j, $s_i s_j \in \{-1, 1\}$. If we substitute (4) into (1), we find that

$$Z = \cosh^M K \cosh^M L \sum_{s \in \mathcal{C}} \prod_{(i,j)} (1 + v s_i s_j) \prod_{(i,k)} (1 + w s_i s_k)$$

where $v = \tanh K$ and $w = \tanh L$. The first product is over all M horizontal edges of the square lattice, whereas the second one is over all the M vertical edges.

The next step is to expand the products appearing in the partition function. In total, the expansion has 2^{2M} terms. It is better to view these terms from the perspective of a graphical calculus as follows:

1. We associate to the factor vs_is_j a line drawn on the horizontal edge (i, j).

2. Similarly, we associate to the factor ws_is_k, a line drawn on the vertical edge (i, k).

3. If there is a factor of 1 instead, no line is drawn.

In this way, we can establish a correspondence between each of the 2^{2M} terms and the graphical configuration on the square lattice created acording to the rules above. Each term takes the form

$$v^r w^s s_1^{n_1} s_2^{n_2} \dots$$

where r and s are the total numbers of horizontal and vertical lines respectively and n_i is the number of lines with lattice point i as its final site. Once, we sum over all the possible spins, we find that we obtain zero unless the numbers $n_1, n_2, \dots n_N$ are all even. In this case, the result of the summation is $2^N v^r w^s$. Therefore, (1) takes the form

$$Z = 2^N (\cosh K \cosh L)^M \sum_P v^r w^s$$

where the sum is over all line configuration on the square lattice with an even number of lines at each site. But, this is just the same as summing over all closed polygons P on the lattice. Also, N is the number of lattice sites on the square lattice. If we define the geometrical quantity

$$\Psi(v, w) = \sum_P v^r w^s,$$

then a very similar argument to the low temperature regime allows us to write out the first few terms of Ψ as

$$\Psi(v, w) = 1 + Nv^2w^2 + N(v^4w^2 + v^2w^4) + \dots.$$

0.4.4 Finding the critical temperature

We notice that the low temperature and high temperature expansions of the partition function have one common feature: they contain a sum over all closed polygonal configurations P on the dual lattice and the original lattice respectively. For finite lattices, the dual lattice and the original lattice only differ at the boundary. In the thermodynamic limit, this difference disappears and consequently, $\Phi(K, L)$ and $\Psi(v, w)$ become related by a change of variables. This suggests we look at the free energy per site, which is once again given by

$$f = \lim_{n\to\infty} \frac{\log Z}{N}.$$

As $N \to \infty$, $M/N \sim 1$ and thus, we have two different expressions for f, given by

$$\begin{aligned} f &= \lim_{N\to\infty} \frac{\log(2^N(\cosh K \cosh L)^M \sum_P v^r w^s)}{N} \\ &= \lim_{N\to\infty} \frac{1}{N}(N\log 2 + M\log(\cosh K \cosh L)) + \frac{\log \sum_P v^r w^s}{N} \\ &= \log(2\cosh K \cosh L) + \lim_{N\to\infty} \frac{\log \sum_P v^r w^s}{N} \end{aligned}$$

and

$$\begin{aligned} f &= \lim_{N\to\infty} \frac{\log(2\exp(M(K+L))\sum_P \exp(-2Lr - 2Ks))}{N} \\ &= \lim_{N\to\infty} \frac{\log 2}{N} + \frac{M(K+L)}{N} + \frac{\log \sum_P \exp(-2Lr - 2Ks)}{N} \\ &= K + L + \lim_{N\to\infty} \frac{\log \sum_P \exp(-2Lr - 2Ks)}{N}. \end{aligned}$$

If we let

$$\rho(v, w) = \lim_{N\to\infty} \frac{\log \sum_P v^r w^s}{N},$$

then

$$f(K, L) = K + L + \rho(e^{-2L}, e^{-2K}) = \log(2\cosh K \cosh L) + \rho(v, w).$$

In the latter equation, we replace K and L with new coupling constants $\tilde{K}$ and $\tilde{L}$, which satisfy

$$\tanh \tilde{K} = -e^{2L} \text{ and } \tanh \tilde{L} = e^{-2K}, \tag{5}$$

then

$$\rho(v, w) = \rho(\tanh \tilde{K}, \tanh \tilde{L}) = \rho(e^{-2L}, e^{-2K})$$

By eliminating the ρ term, we find that

$$-f(\tilde{K}, \tilde{L}) = K + L - f(K, L) - \log(2\cosh \tilde{K} \cosh \tilde{L}).$$

We have related the free energy at low temperature to the free energy at high temperature, via the duality relation purported in (5). We can express (5) in a more symmetric form via the computation below

$$\begin{aligned}\sinh(2K)\sinh(2\tilde{L}) &= 4\sinh(K)\cosh(K)\sinh(\tilde{L})\cosh(\tilde{L}) \\ &= \sinh(K)\cosh(K)(2\sinh(\tilde{L}))(2\cosh(\tilde{L})) \\ &= \sinh(K)\cosh(K)(e^{-K})(e^{K}) \\ &= (e^{-\tilde{L}})(e^{\tilde{L}})(e^{K})(e^{-K}) \\ &= 1\end{aligned}$$

which reveals that

$$\sinh(2K)\sinh(2\tilde{L}) = \sinh(2L)\sinh(2\tilde{K}) = 1 \tag{6}$$

In order to locate the critical point of the square lattice Ising model, suppose that the low temperature Ising model has a line of critical points. The mapping of the coupling constants $(K, L) \mapsto (\tilde{K}, \tilde{L})$ maps the low temperature Ising model to a high temperature one and vice versa, while leaving the curve $\sinh 2K \sinh 2L = 1$ unchanged. To see why this is the case, note that from (6),

$$\sinh 2K \sinh 2L = \frac{1}{\sinh 2\tilde{K} \sinh 2\tilde{L}}.$$

Thus, the only way the map $(K, L) \mapsto (\tilde{K}, \tilde{L})$ does not change is when $\sinh 2K \sinh 2L = 1$. Under this map, we map our critical line to another critical line in terms of $\tilde{K}$ and $\tilde{L}$.

If we make the important assumption that there is only one critical line for the square lattice Ising model, then the critical line must be defined by the equation

$$\sinh 2K \sinh 2L = 1.$$

Now if we assume an isotropic model where $K = L$. Then, if we assume that our model contains only one critical point K_c, then it must satisfy

$$e^{-2K_c} = \tanh 2\tilde{K}_c = \tanh K_c.$$

Solving this equation yields a critical point of $K_c = \frac{1}{2}\log(1 + \sqrt{2})$, which turns out to be the correct critical point of the square lattice Ising model.

0.5 The star-triangle relation

0.5.1 Honeycomb-Triangle Duality

We can apply the duality method depicted for the square lattice Ising model in order to describe the duality between a honeycomb lattice Ising model and a triangular lattice Ising model. The method is very similar to the square lattice.

The honeycomb lattice we are interested in consists of lattice points arranged in hexagons. This time, there are three types of vertices to look out for:

1. The first type of hexagon vertices range from SW to NE. They have coupling constant L_1.
2. The second type of hexagon vertices range from NW to SE. They have coupling constant L_2
3. The third type of hexagon vertices are vertical. They have coupling constant L_3.

Our notation is adopted from [Bax89, Figure 6.5]. The triangle lattice is created out of sites situated at the centre of each hexagon. In this manner, the triangle lattice is dual to the honeycomb lattice. Interestingly, the dual of a honeycomb lattice of $2N$ lattice sites is a triangular lattice of N sites. Similarly, there are three types of vertices to watch out for:

1. The first type of triangle vertices range from NW to SE. They have coupling constant K_1.
2. The second type of triangle vertices range from SW to NE. They have coupling constant K_2
3. The third type of triangle vertices are horizontal. They have coupling constant K_3.

The low temperature expansion of the partition function for the honeycomb lattice with $2N$ sites is given by

$$Z_{2N}^{H}(L) = \exp(N(L_1 + L_2 + L_3)) \sum_{P} \exp(-2L_1 r_1 - 2L_2 r_2 - 2L_3 r_3) \quad (7)$$

Once again, the sum is over all closed polygons P on the honeycomb lattice. Notice that we replaced the expected $2N$ with N in the exponential and removed the leading factor of 2 in (7). This is because the number of edges of each class is now taken to be N, rather than the number of lattice sites. This ignores boundary effects and thus, the replacements we made will not affect the free energy in the thermodynamic limit as $N \to \infty$. Moreover, the number r_i is the number of edges of type i for all $i \in \{1, 2, 3\}$.

Next, we apply the high temperature procedure to expand the partition function for the triangular lattice with N sites. The result is

$$Z_N^T(K) = (2 \cosh K_1 \cosh K_2 \cosh K_3)^N \sum_P v_1^{r_1} v_2^{r_2} v_3^{r_3} \tag{8}$$

where $v_i = \tanh K_i$ for all $i \in \{1, 2, 3\}$. We will now directly compare (7) and (8). Let us make the substitution

$$\tanh K_j^* = e^{-2L_j} \text{ for all } j \in \{1, 2, 3\}$$

in (8) to obtain

$$\begin{aligned} Z_{2N}^H(L) &= e^{N(L_1+L_2+L_3)} \sum_P e^{-2L_1 r_1} e^{-2L_2 r_2} e^{-2L_3 r_3} \\ &= e^{N(L_1+L_2+L_3)} \sum_P (\tanh K_1)^{r_1} (\tanh K_2)^{r_2} (\tanh K_3)^{r_3} \\ &= e^{NL_1} e^{NL_2} e^{NL_3} \sum_P v_1^{r_1} v_2^{r_2} v_3^{r_3}. \end{aligned}$$

To proceed any further, we require the following expressions for all $i \in \{1, 2, 3\}$. These result from straightforward manipulations of hyperbolic identities:

$$\frac{1}{\sinh K_i^*} = e^{2L_i} \sqrt{1 - e^{-4L_i}} \text{ and } \cosh K_i^* = \frac{1}{\sqrt{1 - e^{-4L_i}}}.$$

Continuing the calculation, we have

$$
\begin{aligned}
Z_{2N}^H(L) &= e^{NL_1}e^{NL_2}e^{NL_3}\sum_P v_1^{r_1}v_2^{r_2}v_3^{r_3} \\
&= (e^{2L_1}e^{2L_2}e^{2L_3})^{N/2}\sum_P v_1^{r_1}v_2^{r_2}v_3^{r_3} \\
&= (\frac{\cosh K_1^*}{\sinh K_1^*}\frac{\cosh K_2^*}{\sinh K_2^*}\frac{\cosh K_3^*}{\sinh K_3^*})^{N/2}\sum_P v_1^{r_1}v_2^{r_2}v_3^{r_3} \\
&= \frac{(\cosh K_1^*\cosh K_2^*\cosh K_3^*)^N}{(\sinh K_1^*\cosh K_1^*\sinh K_2^*\cosh K_2^*\sinh K_3^*\cosh K_3^*)^{N/2}}\sum_P v_1^{r_1}v_2^{r_2}v_3^{r_3} \\
&= \frac{2^{3N/2}(\cosh K_1^*\cosh K_2^*\cosh K_3^*)^N}{(\sinh 2K_1^*\sinh 2K_2^*\sinh 2K_3^*)^{N/2}}\sum_P v_1^{r_1}v_2^{r_2}v_3^{r_3} \\
&= \frac{2^{N/2}(2\cosh K_1^*\cosh K_2^*\cosh K_3^*)^N}{(\sinh 2K_1^*\sinh 2K_2^*\sinh 2K_3^*)^{N/2}}\sum_P v_1^{r_1}v_2^{r_2}v_3^{r_3} \\
&= (2a_1a_2a_3)^{N/2}Z_N^T(K^*).
\end{aligned}
$$

Here $a_i = 1/\sinh 2K_i^* = \sinh 2L_i$ for all $i \in \{1,2,3\}$. This reveals the following duality relation between the coupling constants L_i and K_i^*:

$$\sinh 2L_i \sinh 2K_i^* = 1.$$

However, this relation is markedly less useful than (6) for square lattices. This is because the related coupling constants are for two Ising models with different lattice structure. What we really want is a self-duality relation — a duality relation between Ising models with the same lattice structure. We will still be able to determine the critical temperature of both the honeycomb lattice and the triangle lattice. The key ingredient is the **star-triangle relation**.

0.5.2 Star-Triangle Identity

The important observation which belays the star-triangle relation is the fact that the honeycomb lattice is bipartite. As a result, we can divide each lattice site into two classes — A sites and B sites. Additionally, we can do this in such a way that the neighbours of all A sites are B sites and vice versa. See [Mus10, Figure 4.9].

So, the partition function for the honeycomb lattice can be computed by summing over all the B sites first and then the A sites, which yields

$$Z_N^H(L) = \sum_{\sigma_A} \prod_{\langle i,j,k \rangle} w(\sigma_i, \sigma_j, \sigma_k)$$

where

$$w(\sigma_i, \sigma_j, \sigma_k) = \sum_{\sigma_B \in \{-1,1\}} W(\sigma_B; \sigma_i, \sigma_j, \sigma_k) = 2\cosh(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k)$$

and

$$W(\sigma_B; \sigma_i, \sigma_j, \sigma_k) = \exp(\sigma_B(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k)).$$

We are just summing over all B sites first. The key is that due to the fact that $\cosh(-x) = \cosh(x)$ and $\sinh(-x) = -\sinh(x)$, we can write $w(\sigma_i, \sigma_j, \sigma_k)$ in such a way that it is proportional to the Boltzmann factor of the triangular lattice. So, there exists parameters K_1, K_2, K_3 and a constant $\mathcal{D}$ such that

$$w(\sigma_i, \sigma_j, \sigma_k) = \mathcal{D}\exp(K_1\sigma_j\sigma_k + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_j). \tag{9}$$

Our partition function now simplifies to

$$Z_N^H(L) = \mathcal{D}^{N/2} \sum_{\sigma_A} \prod_{\langle i,j,k \rangle} \exp(K_1\sigma_j\sigma_k + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_j).$$

By comparing this expression to the partition function for the triangular lattice, we find that $Z_{2N}^H(L) = \mathcal{D}^N Z_N^T(K)$. It remains to determine how $\mathcal{D}, K_1, K_2$ and K_3 are related to the original coupling constants on the honeycomb lattice L_1, L_2 and L_3.

By substituting all possible values of $\sigma_i, \sigma_j, \sigma_k \in \{-1, 1\}$ into (9), we obtain four distinct equations

$$2\cosh(L_1 + L_2 + L_3) = R\exp(K_1 + K_2 + K_3)$$

$$2\cosh(-L_1 + L_2 + L_3) = R\exp(K_1 - K_2 - K_3)$$

$$2\cosh(L_1 - L_2 + L_3) = R\exp(-K_1 + K_2 - K_3)$$

and

$$2\cosh(L_1 + L_2 - L_3) = R\exp(-K_1 - K_2 + K_3).$$

We will suppress the details of solving the four equations simultaneously. See [Bax89, Pages 82-83] for the details. In summary, we find that for all $i \in \{1, 2, 3\}$,

$$\sinh 2L_i \sinh 2K_i = h^{-1} \tag{10}$$

and

$$\mathcal{D}^2 = 2h \sinh 2L_1 \sinh 2L_2 \sinh 2L_3 \tag{11}$$

where

$$h = \frac{(1 - v_1^2)(1 - v_2^2)(1 - v_3^2)}{4[(1 + v_1v_2v_3)(v_1 + v_2v_3)(v_2 + v_3v_1)(v_3 + v_1v_2)]^{1/2}} \tag{12}$$

where $v_i = \tanh K_i$. Equation (10) tells us that all products of the form $\sinh K_i \sinh L_i$ have the same value.

0.5.3 Critical temperatures of honeycomb and triangular lattices

As an application of the star-triangle identity developed in the previous section, we will compute the critical temperatures of the honeycomb and triangle lattice Ising models. Let us begin with the relationship between the partition functions of these models derived in the previous section:

$$Z_{2N}^H(L) = \mathcal{D}^N Z_N^T(K).$$

But this reveals that

$$(2a_1a_2a_3)^{N/2} Z_N^T(K^*) = \mathcal{D}^N Z_N^T(K).$$

Recall that $a_i = 1/\sinh K_i^*$ for all $i \in \{1, 2, 3\}$. Using (11), we can express the ratio $(2a_1a_2a_3)^{N/2}/\mathcal{D}^N$ as follows:

$$\begin{aligned}
\frac{(2a_1a_2a_3)^{N/2}}{\mathcal{D}^N} &= \frac{(2 \sinh^{-1} 2K_1^* \sinh^{-1} 2K_2^* \sinh^{-1} 2K_3^*)^{N/2}}{(2h \sinh 2L_1 \sinh 2L_2 \sinh 2L_3)^{N/2}} \\
&= \frac{(2 \sinh^{-1} 2K_1^* \sinh^{-1} 2K_2^* \sinh^{-1} 2K_3^*)^{N/2}}{(2h^{-2} \sinh^{-1} 2K_1 \sinh^{-1} 2K_2 \sinh^{-1} 2K_3)^{N/2}} \\
&= (h^{-1})^{N/2} = h^{-N/2}.
\end{aligned}$$

So,

$$h^{-N/2} Z_N^T(K^*) = Z_N^T(K) \tag{13}$$

where for all $i \in \{1, 2, 3\}$, $\sinh 2K_i^* = h \sinh 2K_i$. This time, we have a self-duality relation between two triangle lattice Ising models. We can now employ a similar strategy to the square lattice Ising model in order to obtain the critical temperature of the triangular Ising model. Let us assume that $K_1 = K_2 = K_3 = K$ for simplicity so that $v_1 = v_2 = v_3 = v$. Similarly to the square lattice Ising model, the critical temperature occurs when $h = 1$. This is equivalent to

$$\frac{(1 - v^2)^3}{4[v^3(1 + v^3)(1 + v)^3]^{1/2}} = 1.$$

Squaring both sides of the above equation and simplifying, we obtain

$$(1 + v)^4(1 + v^2)^3(v^2 - 4v + 1) = 0.$$

The only solution which is reasonable and has a physical meaning is $v_c = 2 - \sqrt{3}$. This gives us the critical temperature as $K_c = \operatorname{arctanh}(2 - \sqrt{3})$.

In order to determine the critical temperature of the (isotropic) honeycomb lattice L_c, we can use the relation $\tanh K_c = e^{-2L_c}$. After some computation, we deduce that

$$L_c = \frac{1}{2} \log(2 + \sqrt{3}).$$

Let us compare our critical values for the isotropic honeycomb, square and triangle lattices. Here, we will use the term temperature in the literal sense, recalling that coupling constants scale as $\beta = T^{-1}$.

It turns out that the triangular lattice has the highest critical temperature, followed by the square lattice and then the honeycomb lattice. As [Mus10, Page 160] explains, this is due to coordination numbers (the number of neighbours associated to each spin). The triangular lattice has coordinate number $z = 6$, the square lattice has $z = 4$ and the honeycomb lattice has $z = 3$. So, the larger number of interactions amongst the spins of the triangular lattice results in a system which magnetises at a higher temperature. Physically, more energy is required to overcome the numerous spin interactions in the triangular lattice in order to flip the magnetisation on or off.

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